Polytopic Invariant Set Synthesis

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Danylo Malyuta*, Dylan Janak**, Behçet Açıkmeşe

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Autonomous Controls Laboratory, University of Washington

* danylo@uw.edu
** dylan.janak@gmail.com

Overview

Motivation

Positively Invariant Sets

Polytopes

Minimal Invariant Polytope Construction

Robust Invariant Set Definitions

Minimal Robust Positively Invariant Set Computation

Maximal Controlled Robust Positively Invariant Set Computation (Independent Noise)

Maximal Controlled Robust Positively Invariant Set Computation (Dependent Noise)

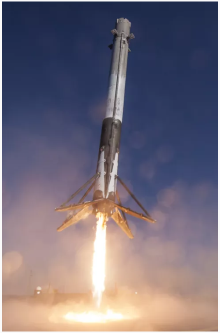
Conclusion

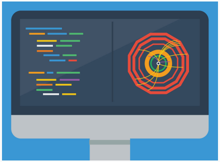
Why Invariance?

In **control**:

- Sets appear naturally in three aspects: constraints, uncertainties, design specifications
- Sets naturally describe system performance (domain of attraction, accuracy, etc.)
- Connection to Lyapunov theory
- Invariance is key in e.g. Model Predictive Control to guarantee resolvability

In **computer science**: applications in optimizing compilers, design by contract, formal methods, program correctness assurance.





Introduction: Robustness to Uncertainty

System model:

$$x^+ = f(x, u, w).$$

The task is to find/verify a control law u which safely brings the state x to the terminal set $\mathcal{X}_T$, in the presence of disturbance w.

How could we ensure that:

- state and control constraints are never violated?
- the state x eventually reaches $\mathcal{X}_T$?
- the state will always stay in $\mathcal{X}_T$ for all time $t \geq T$?

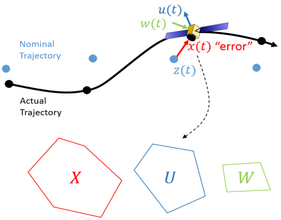
This is impossible in general, since the disturbance may be unbounded.

Tube MPC Approach

Tube MPC separates this problem into two parts:

- Design a nominal trajectory $(\bar{x}(t), \bar{u}(t))$ which brings the state to the set $\bar{\mathcal{X}}_T \subset \mathcal{X}_T$
- Design a feedback law $u(t, x) = \bar{u}(t) + v\big(x(t) - \bar{x}(t)\big)$ which keeps $x(t)$ close to $\bar{x}(t)$.

Invariance analysis can guarantee that $x(t)$ remains in some known set containing $\bar{x}(t)$ for all disturbances $w(t) \in \mathcal{W}$.



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Invariant Sets

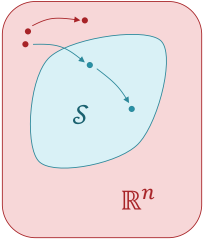
A set $\mathcal{S}$ is **(positively) invariant** with respect to dynamics $x^+ = f(x, w)$ if

$$(x, w) \in \mathcal{S} \times \mathcal{W} \Rightarrow x^+ \in \mathcal{S}.$$

- Trivial examples: $\emptyset$, $\mathbb{R}^n$

For any two invariant sets $\mathcal{S}_1$ and $\mathcal{S}_2$, both $\mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2$ are invariant.

The **Minimal** and **Maximal** invariant sets are respectively the **intersection** and **union** of all invariant sets.



Minimal Invariant Set Computation

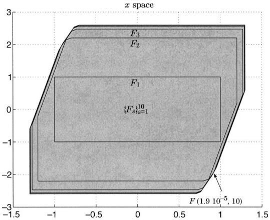
The set of states reachable in finite time from the origin is the minimal positively invariant set, among all sets containing the origin.

Even in the simple linear system $x^+ = Ax + w, w \in \mathcal{W}$, this set is usually difficult to compute exactly, unless:

- $\mathcal{W}$ is convex, and
- $A^s = \alpha I$ for some $\alpha \in [0,1)$ and $s \in \mathbb{N}$.

$$\Rightarrow \mathcal{F}_\infty = \frac{1}{1-\alpha}(\mathcal{W} \oplus A\mathcal{W} \oplus \cdots \oplus A^{s-1}),$$

where $\mathcal{A} \oplus \mathcal{B} = \{a + b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$.



Source: Racović et al.

S. V. Raković, E. C. Kerrigan, K. I. Kouramas, and D. Q. Mayne. *Invariant approximations of the minimal robust positively invariant set.* IEEE Transactions on Automatic Control, 50(3):406–410, March 2005.

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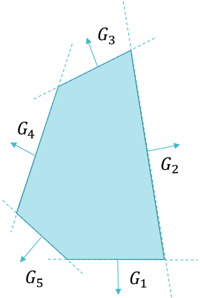
Polyhedra and Polytopes

A **polyhedron** is the intersection of a finite number of closed half-spaces, and can be represented as $\mathcal{P}(G, g) = \{x \in \mathbb{R}^n \mid Gx \leq g\}$.

- The matrix G is called the **template** for a polyhedron.

A **polytope** is a bounded polyhedron, and can also be represented as the convex hull of a finite set of vertices $\{v_1, \dots, v_N\}$; $\mathcal{P}(V) = \{\sum_i v_i\lambda_i \mid \sum_i \lambda_i = 1, \lambda_i \geq 0\}$.



Lattices

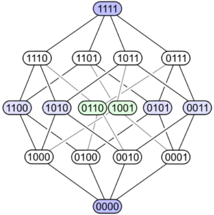
A **poset** $(\mathcal{S}, \leq)$ is a set $\mathcal{S}$ with partial order $\leq$.

A **lattice** is a poset $(\mathcal{S}, \leq)$ which for any $s_1, s_2 \in \mathcal{S}$, there exist a greatest lower bound (meet) $s_1 \wedge s_2 \in \mathcal{S}$ and a least upper bound (join) $s_1 \vee s_2 \in \mathcal{S}$.

- Vectors in $\mathbb{R}^n$ $(\mathbb{R}^n, \leq)$
  - $x \leq y$ if $x(i) \leq y(i), \forall i = 1, \ldots, n$
  - $x \wedge y$ is the elementwise minimum of x and y, and $x \vee y$ is the elementwise maximum

A **complete lattice** is a poset which has a glb and lub for any subset of $\mathcal{S}$.

- Extended real number line $(\mathbb{R} \cup \{+\infty, -\infty\}, \leq)$
- Subsets of $\mathbb{R}^n$ $\left(2^{\mathbb{R}^n}, \subseteq\right)$
  - $\mathcal{X} \subseteq \mathcal{Y}$ if $x \in \mathcal{X} \Rightarrow x \in \mathcal{Y}$
  - $\inf\{\mathcal{X}, \mathcal{Y}\} = \mathcal{X} \cap \mathcal{Y}$, and $\sup\{\mathcal{X}, \mathcal{Y}\} = \mathcal{X} \cup \mathcal{Y}$



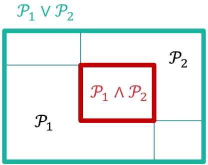
Lattice of Polytopes

For any two polytopes with the same template G, $\mathcal{P}(G, g_1)$ and $\mathcal{P}(G, g_2)$, their intersection is also a polytope with template G. Similarly, there is a unique minimal polytope with template G containing both $\mathcal{P}(G, g_1)$ and $\mathcal{P}(G, g_2)$.

The empty set $\emptyset$ can be represented with any template as $\mathcal{P}(G, -\infty)$.

The set of polytopes with template G that are bounded above by some given nonempty polytope $\mathcal{P}(G, g_{max})$ form a complete lattice.

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$



Knaster-Tarski Theorem

Let $(\mathcal{S}, \leq)$ be a complete lattice, and let $\mathcal{F}: \mathcal{S} \rightarrow \mathcal{S}$ be monotone (order-preserving). Then the set of fixed points of $\mathcal{F}$ also form a complete lattice.

- Corollary: There exist a unique maximal and minimal fixed point of $\mathcal{F}$.

For the lattice $\left(2^{\mathbb{R}^n}, \subseteq\right)$, let $\mathcal{R}(\mathcal{X}) = \{ f(x, w) \mid x \in \mathcal{X}, w \in \mathcal{W} \}$, and $\mathcal{F}(\mathcal{X}) = \mathcal{X} \cup \mathcal{R}(\mathcal{X})$. The following three conditions are equivalent:

- $\mathcal{X}$ is positively invariant w.r.t. the dynamics $x^+ = f(x, w), w \in \mathcal{W}$.
- $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{X}$.
- $\mathcal{X}$ is a fixed point of $\mathcal{F}$.

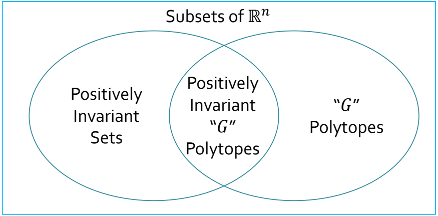
The set of positively invariant sets forms a complete lattice, since it is the set of fixed points for the monotone mapping $\mathcal{F}$.

Minimal and Maximal Invariant Polytopes

The sublattice of polytopes with template G in any nonempty interval $[\mathcal{P}(G, g_l), \mathcal{P}(G, g_u)]$ form a complete lattice.

By the Knaster-Tarski theorem, the set of positively invariant sets form a complete lattice (if nonempty).

Therefore if there is positively invariant $\mathcal{P}(G, g_{inv}) \in [\mathcal{P}(G, g_l), \mathcal{P}(G, g_u)]$, then the positively invariant polytopes with template G in the interval $[\mathcal{P}(G, g_l), \mathcal{P}(G, g_{inv})]$ form a complete lattice.



Sublattices of $(\mathbb{R}^n, \subseteq)$

N.S.C. for Invariance

Let $x^+ = Ax + Ew$, $Fw \leq f$, and $Gx \leq g$ for nonnegative vectors f and g. Then $Gx^+ \leq g$ iff there exist nonnegative matrices Y and M such that:

$$Yg + Mf \leq g,$$
$$YG = GA,$$
$$MF = GE.$$

Observations:

- If $\mathcal{P}(G, g)$ is positively invariant, then
  - $\mathcal{P}(G, \alpha g)$ is positively invariant for all $\alpha \geq 1$.
  - If the disturbance w is scaled by factor $\beta \geq 0$, $\mathcal{P}(G, \beta g)$ is positively invariant.
- If $Yg < g$ and $YG = GA$ for some nonnegative Y, and if w is bounded, then $\mathcal{P}(G, \alpha g)$ is positively invariant for some sufficiently large α.
- These relations are linear in A, E, Y, and M, so the set of all (A, E) making this set positively invariant is itself a polytope in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p}$.

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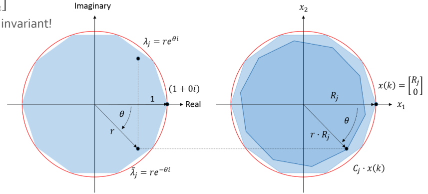
Explicit Template Construction

LTI system with bounded disturbance: $x^+ = Ax + w,\ \|w\| \le d$.

- Real Jordan decomposition $A = V \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_r \end{bmatrix} V^{-1}$, where each C_i is a real Jordan block.
  - **λ_i real**: $C_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$. $\quad$ $\boldsymbol{\lambda_{i,i+1} = \alpha_i \pm j\beta_i}$: $C_i = \begin{bmatrix} \begin{matrix}\alpha_i & \beta_i \\ -\beta_i & \alpha_i\end{matrix} & I & & \\ & \begin{matrix}\alpha_i & \beta_i \\ -\beta_i & \alpha_i\end{matrix} & \ddots & \\ & & \ddots & I \\ & & & \begin{matrix}\alpha_i & \beta_i \\ -\beta_i & \alpha_i\end{matrix} \end{bmatrix}$.
- Change of variables $\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = V^{-1}x \Rightarrow z_i^+ = C_i z_i + \tilde{w}_i$, with $\|\tilde{w}_i\| \le \tilde{d}_i$.
- If λ_i is real, then $\tilde{G}_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Otherwise, $\tilde{G}_{i,i+1}$ defines the regular polygon in the figure.
- $G = diag(g)^{-1} \begin{bmatrix} \tilde{G}_1 & & \\ & \ddots & \\ & & \tilde{G}_n \end{bmatrix} V^{-1}$, with g given explicitly or as a solution to a linear program.
- $\{x \mid \|Gx\|_\infty \le 1\}$ is positively invariant!

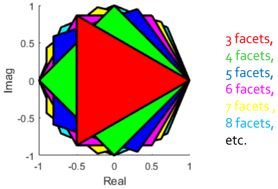


J.C. HENNET, Discrete-time Constrained Linear Systems.Control and Dynamic Systems, Vol.71, C.T. Leondes Ed., Academic Press, pp.157-213.

Explicit Invariant Polytope Construction

Observations:

- With linearized dynamics $x^+ = Ax + Bu + Ew$, and assuming (A, B) is controllable, the eigenvalues of $(A + BK)$ can be placed anywhere in $\mathbb{C}$ with the proper choice of K.
- $(A + BK)$ is diagonalizable if each eigenvalue is unique, and so this polytope can be found numerically via diagonalization.
- This algorithm gives an upper bound for the necessary number of facets of a positively invariant polytopes as a function of the eigenvalue locations:
  - Each real eigenvalue contributes 2 facets.
  - Each pair of complex eigenvalues contributes $m \geq 3$ facets, with the pair of eigenvalues in the interior of a regular m-gon inscribed in the unit circle in the complex plane with a vertex at $1 + \boldsymbol{j}0$.
- Generalizes the "unit diamond" condition [Bitsoris, 1988], which guarantees a positively invariant "box" if all eigenvalues $\lambda = \alpha + \boldsymbol{j}\beta$ satisfy $|\alpha| + |\beta| < 1$, the green square in the figure.



George Bitsoris. Positive invariant polyhedral sets of discrete-time linear systems. *International Journal of Control*, 47(6):1713–1726, 1988.

Minimal Invariant Polytope

Since the set of polyhedra with given template G form a complete lattice, the Knaster-Tarski theorem guarantees a minimal positively invariant polytope in this set. Let $\mathcal{S}$ be a closed and bounded set of polytopes with template G.

<u>Iterative algorithm (Kleene):</u>
If a function f commutes with $\vee$, then the minimal fixed point of f containing set $\mathcal{X}_0$ can be computed as:

$$\mathcal{X}_0 \vee f(\mathcal{X}_0) \vee f\big(f(\mathcal{X}_0)\big) \vee \cdots .$$

Start with an initial set $\mathcal{X}_0 \in \mathcal{S}$. For $k = 0, 1, \ldots$, compute $\mathcal{X}_{k+1} = \mathcal{X}_k \vee \mathcal{R}(\mathcal{X}_k)$ until some convergence criterion is met. $\mathcal{X}_\infty = \bigvee_{k \in \mathbb{N}} \mathcal{X}_k$ is the minimal positively invariant element of $\mathcal{S}$ which contains $\mathcal{X}_0$.

$g^{[0]} \leftarrow 0$
$k \leftarrow 0$

For $k = 0, 1, \ldots, k_{max}$

  $\underset{g^{[k+1]}, x_i, w_i}{\text{maximize}} \; 1^T g^{[k+1]} \; s.t.$

  For $i = 1, \ldots, n$

    $e_i^T g^{[k+1]} \le e_i^T G(Ax_i + Ew_i),$

    $Gx_i \le g^{[k]},$

    $Fw_i \le f.$

  End For

End For

Linear program which computes $\mathcal{R}(\mathcal{X}_k)$ for polyhedral $\mathcal{W}$.

Example

System: discretized harmonic oscillator

$$x^+ = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix} x + \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix} u + w, \|w\|_\infty \le 1.$$

Closing the loop with linear full state feedback from LQR,

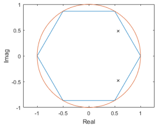
$$x^+ = \begin{bmatrix} 0.791 & 0.445 \\ -0.626 & 0.336 \end{bmatrix} x + w.$$

$A = VCV^{-1}$, with $V = \begin{bmatrix} -0.278 & -0.582 \\ 0.764 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0.564 & 0.477 \\ -0.477 & 0.564 \end{bmatrix}$.

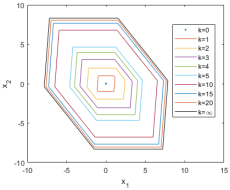
With $m = 6$,

$$G = \begin{bmatrix} 1 & 0 \\ \cos(\pi/3) & \sin(\pi/3) \\ \cos(2\pi/3) & \sin(2\pi/3) \\ \cos(3\pi/3) & \sin(3\pi/3) \\ \cos(4\pi/3) & \sin(4\pi/3) \\ \cos(5\pi/3) & \sin(5\pi/3) \end{bmatrix} V^{-1}.$$

The mapping $\mathcal{X}_{k+1} = \mathcal{X}_k \vee \mathcal{R}(\mathcal{X}_k)$ is performed iteratively starting with $\mathcal{X}_0 = \{0\}$.



The closed-loop eigenvalues $\lambda = 0.564 \pm \boldsymbol{j}0.477$, are in the interior of the regular 6-gon, so 6 facets are sufficient.



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Robust Controlled Invariant Set

We consider a Discrete Linear Time Invariant (DLTI) system:

$$x^{+} = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^n$, $p \in \mathcal{P} = \{p \in \mathbf{R}^d : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^m : Hu \leq h\}$ are "specification" polytopes.

Controlled Robust Positively Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Robust Invariant Set

Robust Controlled Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Now consider that some control law exists and the system reduces to an autonomous one:

$$x^+ = Ax + Dp.$$

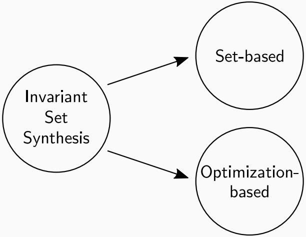
Robust Positively Invariant Set

A set $\mathcal{X}$ is called *robust positively invariant* (RPI) if:

$$Ax + Dp \in \mathcal{X}, \quad \forall x \in \mathcal{X},\ p \in \mathcal{P}.$$

Goal: find an RPI $\mathcal{X}$.

Two Ways to Synthesize an Invariant Set



- Optimization-based methods rely on an explicit optimization problem (LP, LMI, etc.) to find $\mathcal{X}$
- Set-based methods rely on polytopic operations[1], i.e. computational geometry.

[1]These operations may implicitly involve an optimization, but what differentiates set-based methods is that people don't "talk" about it – they just assume that one can compute e.g. the Pontryagin difference.

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One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Equivalent RPI Condition

$$\mathcal{X}(g) = \{x : Gx \leq g\} \text{ RPI} \Leftrightarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P}) \leq \sigma(G_i \mid \mathcal{X}(g)),$$

where $g \in \mathbf{R}^{n_g}$ and $\sigma(z \mid \mathcal{S}) \triangleq \sup\{y^T z : y \in \mathcal{S}\}$ is the support function of (some) set $\mathcal{S}$.

Note: $\sigma(G_i \mid \mathcal{X}(g)) \leq g_i$ with $<$ $\Leftrightarrow$ facet i is redundant.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

Fix G in $\mathcal{X}(g) = \{x : Gx \leq g\}$ (i.e. pick a "template"). Assumptions:

A1. $\mathcal{P}$ contains the origin

A2. $|\lambda| < 1 \ \forall \lambda \in \text{spec}(A)$

A3. The interior of $\mathcal{X}$ contains the origin

A4. For the chosen G, a g exists such that $\mathcal{X}(g)$ is RPI

Then there exists a g^* such that

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

Fixed-Point Solution Uniqueness

Given assumptions A1-A4, the g^* in the above statement is unique.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Existence of an RPI Set

$$\sigma(G_i \mid A\mathcal{X}(g^*)) + \sigma(G_i \mid D\mathcal{P}) = \sigma(G_i \mid \mathcal{X}(g^*)) = g^* \quad \forall i = 1, \ldots, n_g.$$

$\mathcal{X}(g^*)$ is the min-volume RPI set, i.e. g^* achieves minimum $\|g^*\|_1$.

g^* can be computed iteratively:

Algorithm 1 Iterative computation of g^*.

Set $g \leftarrow 0$
while True **do**
  $g_i^* \leftarrow \sigma(G_i \mid A\mathcal{X}(g)) + \sigma(G_i \mid D\mathcal{P})$ $i = 1, \ldots, n_g$
  if $\|g - g^*\|_\infty < \epsilon_{\text{tol}}$ **then**
    return g^*
  $g \leftarrow g^*$

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

- g^* can also be computed as a one-shot LP (main contribution of [1])
- Let $c_i(g) = \sigma(G_i \mid A\mathcal{X}(g))$, $d_i = \sigma(G_i \mid D\mathcal{P})$, $b_i(g) = \sigma(G_i \mid \mathcal{X}(g))$. Core realization (thanks to uniqueness of g^*):

$$g^* = \arg\min_g \{\|g\|_1 : c(g) + d = b(g)\} = \arg\max_g \{\|g\|_1 : c(g) + d = b(g)\}$$

- Recalling that $b(g) \leq g$, the above is readily converted to an LP:

$$g^* = c^* + d^*, \text{ where } (c^*, d^*) = \arg \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\text{maximize}} \quad \sum_{i=1}^{n_g} c_i + d_i$$

$$\text{subject to} \quad c_i \leq G_i A \xi^i, \quad G\xi^i \leq c + d, \quad d_i \leq G_i D \omega^i, \quad F\omega^i \leq g.$$

One-Step Minimal RPI Computation

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Let $c_i(g) = \sigma(G_i \mid A\mathcal{X}(g))$, $d_i = \sigma(G_i \mid D\mathcal{P})$, $b_i(g) = \sigma(G_i \mid \mathcal{X}(g))$

$$
\begin{aligned}
g^* = c^* + d^*, \text{ where } (c^*, d^*) = \; & \underset{\substack{\{c_i, d_i, \xi^i, \omega^i\} \\ \forall i \in \{1,\dots,n_g\}}}{\arg\text{maximize}} && \sum_{i=1}^{n_g} c_i + d_i \\
& \text{subject to} && c_i \leq G_i A \xi^i \\
& && G\xi^i \leq c + d \\
& && d_i \leq G_i D \omega^i \\
& && F\omega^i \leq g.
\end{aligned}
$$

The first two constraints evaluate $c_i(g)$ and the last two evaluate d_i. The first constraint holds with equality at optimality, since we want to maximize c_i. The RHS of the second constraint $= g^*$ at optimality, therefore the second constraint enforces $P\xi^i \leq g^*$, i.e. the definition of $b(g^*)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

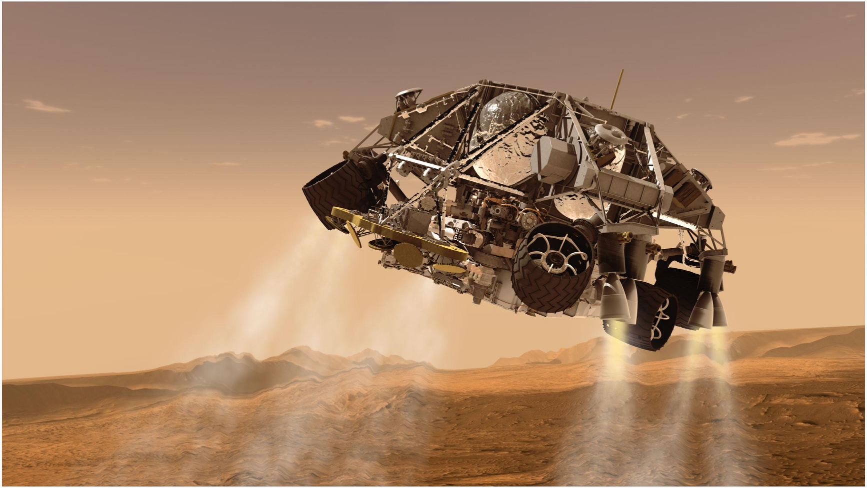
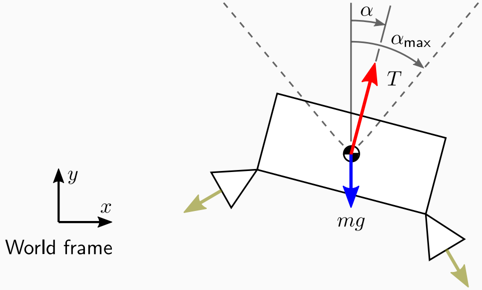


Image credit: NASA/JPL-Caltech

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]



Parameters [2]:

$$
\begin{aligned}
m_{\mathrm{wet}} &= 1905 \text{ kg} \\
g &= -3.7114 \text{ m/s}^2 \\
g_{\mathrm{e}} &= 9.81 \text{ m/s}^2 \\
I_{\mathrm{sp}} &= 225 \text{ s} \quad T_{\max} = 3.1 \text{ kN} \\
\phi &= 27 \text{ deg} \quad n = 6
\end{aligned}
$$

Dynamics:

$$
\begin{aligned}
(\dot{x}, \dot{y}) &= (v_x, v_y) \\
(\dot{v}_x, \dot{v}_y) &= (T_x, T_y)/m + g \\
\dot{m} &= -\frac{\|(T_x, T_y)\|_2}{I_{\mathrm{sp}} g_{\mathrm{e}} \cos\phi}
\end{aligned}
$$

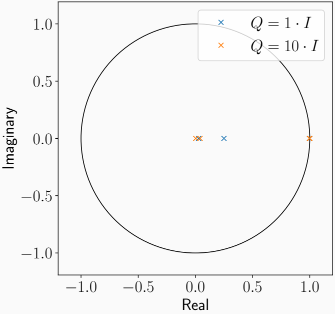
Letting $T \leftarrow T + mg$ be the gravity compensated control, the system is linearized about $(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y, \bar{m}) = (0, 0, 0, 0, m_{\mathrm{wet}})$ and $(\bar{T}_x, \bar{T}_y) = (0, 0)$.

One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

Synthesize an LQR stabilizing controller:

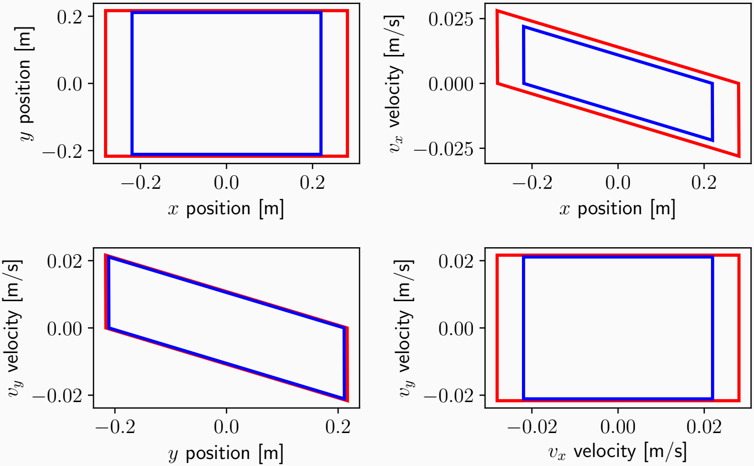
- State scaling: $D_x = \begin{bmatrix} 1 & 1 & 0.05 & 0.05 & 0.1 \end{bmatrix}$
- Input scaling: $D_u = \begin{bmatrix} nT_{\max} \cos\phi \sin\alpha_{\max} & nT_{\max} \cos\phi \end{bmatrix}$
- State penalty $Q = D_x^{-1} \hat{Q} D_x$ with $\hat{Q} \in \{I_5, 10I_5\}$
- Input penalty $R = D_x^{-1} \hat{R} D_x$ with $\hat{R} = I_2$



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

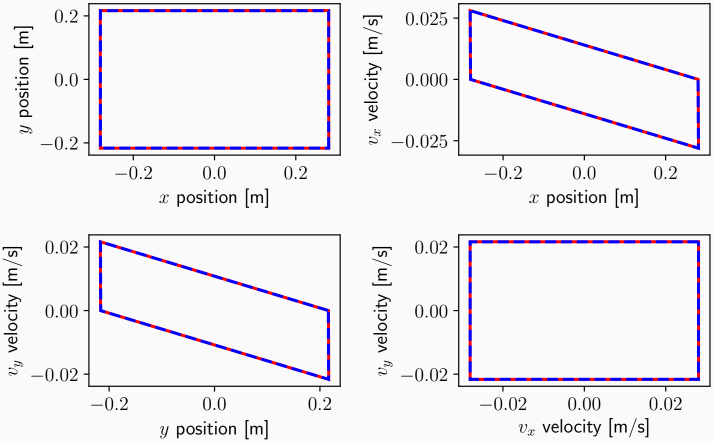
Direct application of LP on slide 29 ($\hat{Q} = I_5$, $\hat{Q} = 10I_5$):



One-Step Minimal RPI Computation

Trodden, "A One-Step Approach to Computing a Polytopic Robust Invariant Set", 2016. [1]

The one-shot LP of slide 29 and the iterative algorithm of slide 27 are identical...



... but iterative takes ≈ 315 s while one-shot takes ≈ 0.2 s!

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Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

We consider Discrete Linear Time Invariant (DLTI) system:

$$x^+ = Ax + Bu + Dp,$$

where $x \in \mathbf{R}^n$, $p \in \mathcal{P} = \{p \in \mathbf{R}^d : Rp \leq r\}$ and $u \in \mathcal{U} = \{u \in \mathbf{R}^m : Hu \leq h\}$ are "specification" polytopes.

Controlled Robust Positively Invariant Set

A set $\mathcal{X}$ is called *controlled robust positively invariant* (CRPI) if:

$$\mathcal{X} = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X},\ \forall p \in \mathcal{P}\}.$$

Maximal CRPI Set

A set $\mathcal{X}_\infty \subseteq \mathcal{X}$ is called *maximal CRPI* (maxCRPI) if it is CRPI and contains all other CRPI sets in $\mathcal{X}$, i.e. $\mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}_\infty\ \forall \mathcal{X}_{\text{CRPI}} \subseteq \mathcal{X}$ RCPI [3].

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

maxCRPI Set Convexity

Given the system $x^+ = Ax + Bu + Dp$ where $p \in \mathcal{P}$, $u \in \mathcal{U}$, consider $\mathcal{X}$ the set of "safe" states. If $\mathcal{X}, \mathcal{P}, \mathcal{U}$ are convex then the associated maxCRPI set $\mathcal{X}_\infty$ is convex.

Recall the maxCRPI set definition:

$$\mathcal{X}_\infty = \{x \in \mathbf{R}^n : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{X}_\infty,\ \forall p \in \mathcal{P}\}.$$

The definition is recursive ($\mathcal{X}_\infty$ on both sides) $\Rightarrow$ compute $\mathcal{X}_\infty$ *iteratively*.

Core step: *preimage set* computation.

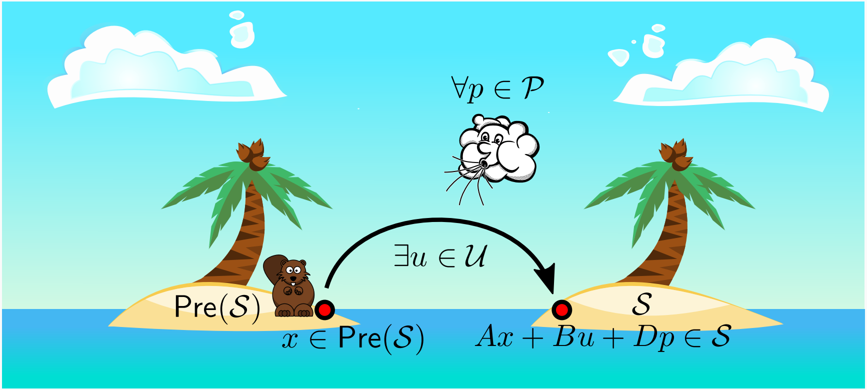
Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set

$$\mathrm{Pre}(\mathcal{S}) \triangleq \{x \mid \exists u \in \mathcal{U},\ Ax + Bu + Dp \in \mathcal{S}\ \forall p \in \mathcal{P}\}$$

Remark: $\mathcal{S}$ CRPI $\Leftrightarrow \mathcal{S} \subseteq \mathrm{Pre}(\mathcal{S})$.



Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

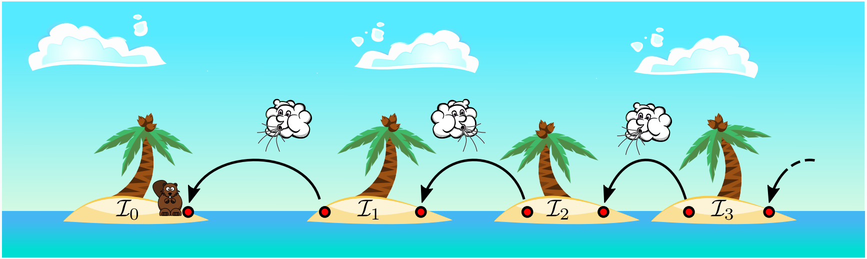
maxCRPI Iterative Computation

Execute the following dynamic programming-type algorithm:

$$\mathcal{I}_0 = \mathcal{X}$$

$$\mathcal{I}_{k+1} = \mathrm{Pre}(\mathcal{I}_k) \cap \mathcal{I}_k \quad k = 0, 1, 2, \ldots$$

STOP if $\mathcal{I}_{k+1} = \mathcal{I}_k$. Then, $\mathcal{I}_k = \mathcal{I}_\infty$ is the maxCRPI set.



(Proxy for convergence: distance between the islands.)

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Preimage Set Computation

$$\mathrm{Pre}(\mathcal{S}) = ((\mathcal{S} \ominus (D\mathcal{P})) \oplus (-B\mathcal{U}))A$$

where[2]:

- Minkowski sum: $\mathcal{A} \oplus \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\}$, $\mathcal{O}(c^n)$
- Pontryagin difference: $\mathcal{A} \ominus \mathcal{B} = \{a : a + b \in \mathcal{A}, \forall b \in \mathcal{B}\}$, $\mathcal{O}(n^c)$
- Direct mapping: $M\mathcal{A} = \{Ma : a \in \mathcal{A}\}$, $\mathcal{O}(c^n)$
- Inverse mapping: $\mathcal{A}M = \{a : Ma \in \mathcal{A}\}$, $\mathcal{O}(n^c)$

Minkowski sum is the most expensive operation (highest facet count, cannot be pre-computed).

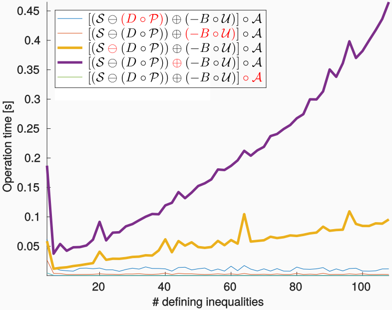
[2] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

$$\text{Pre}(\mathcal{S}) = [(\mathcal{S} \ominus (D \circ \mathcal{P})) \oplus (-B \circ \mathcal{U})] \circ \mathcal{A}$$

For independent disturbances, Pontryagin difference ($\mathcal{O}(n^c)$) and especially Minkowski sum ($\mathcal{O}(c^n)$) are expensive[3].



[3] n is the polytope facet count and c is a coefficient.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

However, may wish to render invariant only *part* of the state. Examples:

- Some states do not make physical sense to render invariant (our case: skycrane mass)
- Some states may correspond to the controller (e.g. integrator)

In this case we want to render invariant the output $y = Cx$.

Controlled Robust Positively Output Invariant Set

The set $\mathcal{X}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } Ax + Bu + Dp \in \mathcal{Y}\ \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Controlled Robust Positively Output Invariant Set

The set $\mathcal{Y}$ is *Controlled Robust Positively Output Invariant* (CRPOI) if:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(Ax + Bu + Dp) \in \mathcal{Y} \; \forall x \text{ s.t. } y = Cx, \forall p \in \mathcal{P}\}$$

Using $C^\dagger$ the pseudoinverse of C, we can write:

$$\mathcal{Y} = \{y : \exists u \in \mathcal{U} \text{ s.t. } C(A(C^\dagger y + \mathcal{N}(C)) + Bu + Dp) \subseteq \mathcal{Y} \; \forall p \in \mathcal{P}\},$$

where $\mathcal{N}(C)$ is the nullspace of C, i.e. $\mathcal{N}(C) = \{x : Cx = 0\}$. The preimage set can be computed similarly to before:

$$\text{Pre}(\mathcal{Y}) = ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$$

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

The following algorithm summarizes maxCRPOI set computation[4].

Algorithm 2 Iterative computation of maxCRPOI set $\mathcal{Y}_\infty$.

Set $\mathcal{Y}$ to the "safe outputs" specification
while True **do**
  $\text{Pre}(\mathcal{Y}) \leftarrow ((\mathcal{Y} \ominus (CD\mathcal{P} \oplus CA\mathcal{N}(C))) \oplus (-CB\mathcal{U}))CAC^\dagger$
  $\mathcal{Y}^+ = \mathcal{Y} \cap \text{Pre}(\mathcal{Y})$
  if $\mathcal{Y} \subseteq \mathcal{Y}^+_{\epsilon_{\text{tol}}}$ and $\mathcal{Y}^+ \subseteq \mathcal{Y}_{\epsilon_{\text{tol}}}$ **then**
    return $\mathcal{Y}_\infty \leftarrow \mathcal{Y}^+$
  $\mathcal{Y} \leftarrow \mathcal{Y}^+$

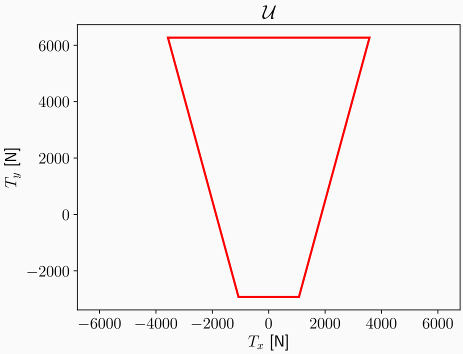
[4]If $\mathcal{S} = \{x : Px \le p\}$, we denote $\mathcal{S}_{\epsilon_{\text{tol}}} = \{x : Px \le p + \epsilon_{\text{tol}}\}$ the ϵ_{tol}-*dilation* of $\mathcal{S}$. In practical, dilation is a more robust stopping criterion than equality ($\mathcal{Y}^+ = \mathcal{Y}$) which is prone to numerical inaccuracy.

Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Going back to the skycrane example, consider the specifications:

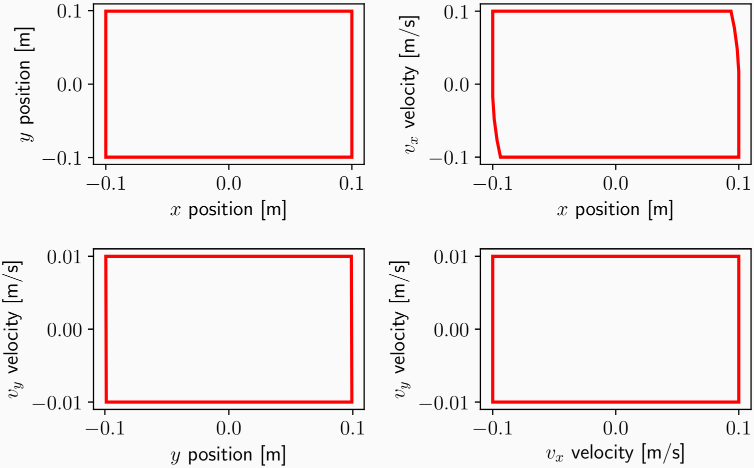
- ± 10 cm position error (in both x and y)
- ± 10 cm/s velocity error in x, ± 1 cm/s velocity error in y
- ± 50 N disturbance force (in both x and y)
- Input constraint set given by the rocket motor specs [2] (visualized below)



Maximal RCI Computation

Kvasnica et al., "Reachability Analysis and Control Synthesis for Uncertain Linear Systems...", 2015. [3]

Direct application of algorithm on slide 43:



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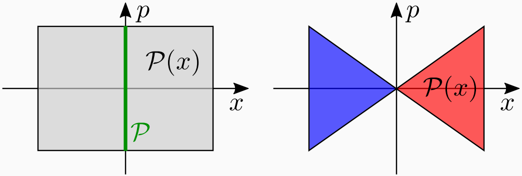
Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

What happens if the disturbance is state and/or input dependent?

$$p \in \mathrm{Proj}_p\,\mathcal{P}(x,u) = \{\theta = (p,x,u) \in \mathbf{R}^{d+n+m} : R\theta \le r\}$$



Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

What happens if the disturbance is state and/or input dependent?

$$p \in \operatorname{Proj}_p \mathcal{P}(x,u) = \{\theta = (p,x,u) \in \mathbf{R}^{d+n+m} : R\theta \leq r\}$$

In this case $\operatorname{Pre}(\mathcal{X})$ can be computed in several steps:

$$\begin{aligned}
\mathcal{Z} &\triangleq \mathcal{X} \times \mathcal{U} \\
\mathcal{W} &\triangleq \{(x,u,p) : (x,u) \in \mathcal{Z}, p \in \mathcal{P}(x,u)\} \\
\Phi &\triangleq \{(x,u,p) : Ax + Bu + Dp \in \mathcal{S}\} \\
\Sigma &\triangleq \{(x,u) \in \mathcal{Z} \mid Ax + Bu + Dp \in \mathcal{S} \ \forall p \in \mathcal{P}(x,u)\} \\
&= \mathcal{Z} \setminus \operatorname{Proj}_{x,u}(\mathcal{W} \setminus \Phi) \\
\Rightarrow \operatorname{Pre}(\mathcal{S}) &= \operatorname{Proj}_x(\Sigma)
\end{aligned}$$

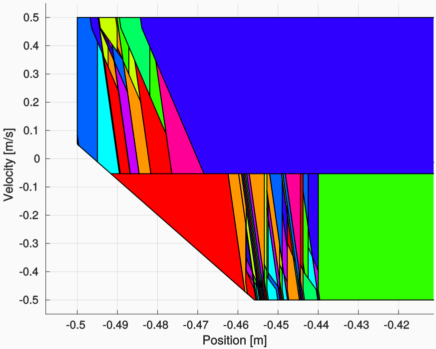
When sets are polytopes, all operations are possible via computational geometry.

Maximal RCI Computation With Dependent Noise

Rakovic et al., "Reachability Analysis of Discrete-Time Systems With Disturbances", 2006. [4]

$$\Sigma = \mathcal{Z} \setminus \text{Proj}_{x,u}(\mathcal{W} \setminus \Phi).$$

Regiondiff operation (\) [5]) generates a union of polytopes, which suffers from severe "fracturing" of convex regions.



Furthermore, $\text{Proj}_{x,u}$ is expensive when $\dim(\mathcal{W})$ is large!

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- Minimal RPI computation boils down to a one-shot LP [1]
- Maximal CRPI computation for generic polytopes has exponential complexity in set-based methods due to the Minkowski sum
- Maximal CRPI computation for dependent noise is computationally difficult due to non-convexity
- Further reading in [6, 7].

Thank You For Your Attention!

Appendix

Overview

Bibliography

Bibliography

[1] P. Trodden, "A one-step approach to computing a polytopic robust positively invariant set," *IEEE Transactions on Automatic Control*, vol. 61, pp. 4100–4105, dec 2016.

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Polyhedral Lyapunov Functions

- Consider the system $x(t+1) = A(t)x(t)$, where $A(t)$ comes from a compact set. The following three statements are equivalent:
  - The system is asymptotically stable
  - The system is exponentially stable
  - The system has a polyhedral Lyapunov function $V(x) = \|Gx\|_\infty$
- Polyhedra are expressive enough to prove exponential stability of a linear system, though it may be difficult to find a suitable "G".
  - For the simple case $x(t+1) = Ax(t)$, a Lyapunov function can be found from the Jordan decomposition of A.
  - With a known quadratic Lyapunov function, a polytope can approximate the ellipsoidal level set arbitrarily well.

Lattices of V-Polytopes

- For each polytope of the form $\mathcal{P} = \{x \in \mathbb{R}^n \mid Gx \leq \mathbf{1}\}$, there is a unique dual polytope $\mathcal{P}^* = \text{co}\{G^T e_1, \dots, G^T e_m\}$ whose interior contains the origin.
- Important properties:
  - For any polytope $\mathcal{P}$ whose interior contains the origin, $\mathcal{P}^{**} = \mathcal{P}$.
  - Order is reversed when taking the dual, i.e. $\mathcal{P}_1 \subseteq \mathcal{P}_2 \Rightarrow \mathcal{P}_2^* \subseteq \mathcal{P}_1^*$.
- Since the set of H-polytopes with template G form a complete lattice, so do their dual polytopes, $\{\text{co}\{\alpha_1 G^T e_1, \dots, \alpha_m G^T e_m\} \mid 0 \leq \alpha_i \leq \alpha_{max}\}$.

$$G^T = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$