FEANICSES

# Mixed Continuous/Discrete Time Reachability Analysis with Integral Quadratic Constraint & Paraboloids

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ONERA – LAAS

May 25rd, 2018

# Outline

# Reachability Analysis

$u : [0, T] \mapsto \mathcal{U}$ and $x_0 \in \mathcal{X}$ given, $x : [0, T] \mapsto \mathcal{X}$ solution of the initial value problem:

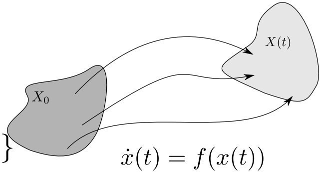
$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

$$\begin{aligned} \Phi : \mathbb{R}^+ \times \mathcal{U}^{[0,T]} \times \mathcal{X} &\mapsto \mathcal{X} \\ \Phi(T, x_0, u) &= x(T) \end{aligned} \tag{2}$$

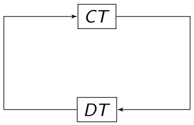
## Reachability analysis

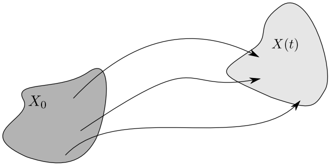
For $X_0 \subset \mathcal{X}$, find an over-approximation of

$$X(T, X_0, u) = \{\Phi(T, x_0, u) | x_0 \in X_0\} \tag{3}$$



# Mixed discrete/continuous time dynamical system



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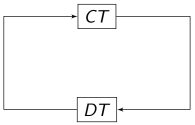


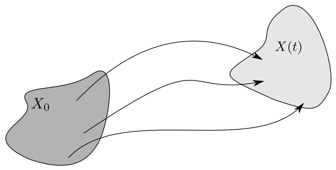


- discrete transitions: $\frac{t}{T_d}$

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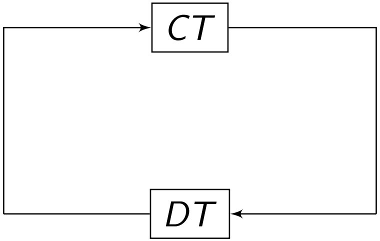

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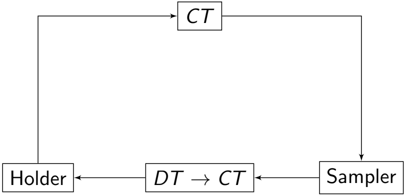

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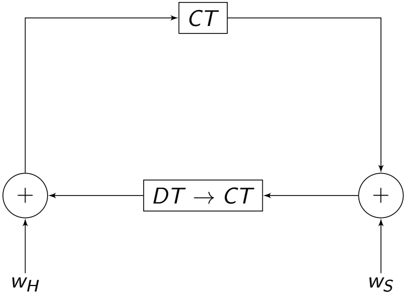


$\Rightarrow$ Constraint on $w_H$ and $w_S$?

# Sampler/Holder Block Abstraction

- Holder/Sampler: modeled as a varying delay

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⇒ for a signal $v : \mathbb{R}^+ \mapsto \mathbb{R}$, $w = \mathit{Sampler}(v)$ defined by

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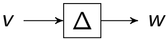
Use energetic inequalities over varying delays to characterize the $w_H$ and the $w_S$.

# Outline

# Integral Quadratic Constraints

- $\Delta$ non linearity
- $G$ linear time invariant
- $x_q$ energetic state $\Rightarrow$ constrain between $v$ and $w$




$\Delta$ verifies the IQC $\Pi$ if for all $v \in \mathcal{L}_2$ and $w = \Delta(v)$:

$$\int_0^{\infty} \begin{bmatrix} \tilde{v}(j\omega) \\ \tilde{w}(j\omega) \end{bmatrix}^\top \Pi(j\omega) \begin{bmatrix} \tilde{v}(j\omega) \\ \tilde{w}(j\omega) \end{bmatrix} d\omega \geq 0$$

Temporal form:

$$\sigma = \begin{bmatrix} x \\ v \\ w \end{bmatrix}^\top M \begin{bmatrix} x \\ v \\ w \end{bmatrix}, \quad \int_0^{\infty} \sigma(t) dt \geq 0, \quad \dot{x} = Ax + B \begin{bmatrix} v \\ w \end{bmatrix}$$

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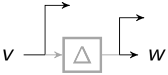


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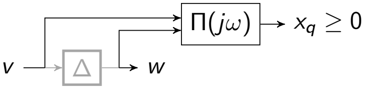


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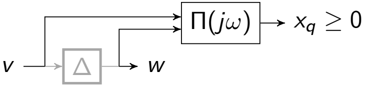


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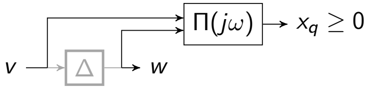


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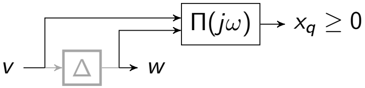


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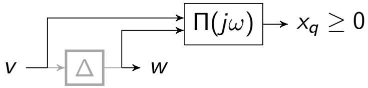


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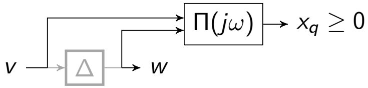


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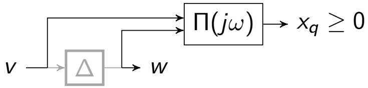


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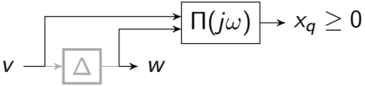
## IQC modelling Example




| Δ | système avec retard | saturation |
|---|---|---|
| |   delay ($\tau \leq T_d$) Holder/Sampler $\|w - v\|_2 < \|\Phi_{T_d}(s)v\|_2$ |   $\forall t > 0, v^2 < t^2 \Rightarrow (v - w)(tw - v) > 0$ |

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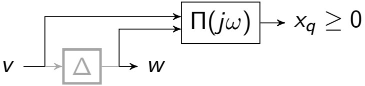


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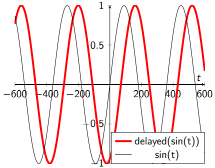
| |  |  |
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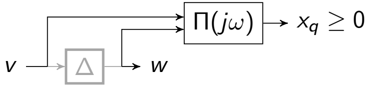


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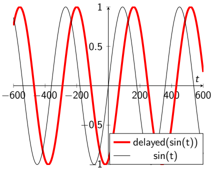
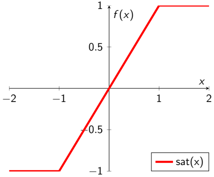
| |   delay $(\tau \leq T_d)$<br>Holder/Sampler<br>$\|w - v\|_2 < \|\Phi_{T_d}(s)v\|_2$ |   $\forall t > 0, v^2 < t^2 \Rightarrow (v - w)(tw - v) > 0$ |

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## IQC origins

- Introduced by Rantzer (1980) study stability of nonlinear systems [1]
- Kalman-Yakubovich-Popov lemma ⇒ semi definite optimisation problem

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## Complete IQC

Minimax theorem $\Rightarrow$ some IQC are Complete IQC [2]:

### Complete IQC

$$\begin{cases} \int_0^\infty \sigma(t)dt \geq 0 \\ \begin{bmatrix} I_x \\ 0 \\ 0 \end{bmatrix}^\top M \begin{bmatrix} I_x \\ 0 \\ 0 \end{bmatrix} > 0 \\ \begin{bmatrix} 0 \\ 0 \\ I_w \end{bmatrix}^\top M \begin{bmatrix} 0 \\ 0 \\ I_w \end{bmatrix} < 0 \end{cases} \Rightarrow \forall T \geq 0, \int_0^T \sigma(t)dt \geq 0$$

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## Complete IQC

$$\int_0^{T+dt} \sigma(t)dt \geq 0$$

$$\int_0^{T} \sigma(t)dt + \sigma(T)dt \geq 0$$

$$\int_0^{T} \sigma(t)dt \geq -\sigma(T)$$

$M_w = \begin{bmatrix} 0 \\ 0 \\ I_w \end{bmatrix}^\top M \begin{bmatrix} 0 \\ 0 \\ I_w \end{bmatrix}$, as $M_w < 0$:

$$\|w\|_P^2 \, dt \leq C\left(\int_0^T \sigma(t)dt, x, v, dt\right)$$

with $P = -M_w > 0$.

$\Rightarrow$ $w$ is bounded!

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# Reachability Analysis for Linear system with Ellipsoids

Ellipsoid of radius $Q \in R^{n,n}$, $Q > 0$ and centered on $x_c \in \mathbb{R}^n$:

$$\mathcal{E}(Q, x_c) = \{x \in \mathbb{R}^n | (x - x_c)^\top Q^{-1}(x - x_c) \leq 1\} \qquad (4)$$




Find the functions $Q : [0, T] \mapsto \mathbb{R}^{n,n}$ and $x_c : [0, T] \mapsto \mathbb{R}^{n,n}$ such that $\forall t > 0, X(t, X_0, U) \subset \mathcal{E}(Q(t), x_c(t))$ [3]

$$\begin{aligned} \dot{Q} &= AQ + QA^\top + hQ + h^{-1}BRB^\top & P(0) &= P_0 \\ \dot{x}_c &= Ax_c + Bu & x_c(0) &= x_{c0} \\ h &= \sqrt{n^{-1}Tr(Q^{-1}BRB^\top)} \end{aligned} \qquad (5)$$

[3] Chernousko F.L. (1999) *What is Ellipsoidal Modelling and How to Use It for Control and State Estimation?* In: Elishakoff I. (eds) Whys and Hows in Uncertainty Modelling. CISM Courses and Lectures, vol 388. Springer, Vienna

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$u(t) \in U = \mathcal{E}(R, u_c)$ $\xrightarrow{u(t)}$ [ $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) \in \mathcal{E}(Q_0, x_{c0})$ ] $\xrightarrow{x(t)}$ $X(t, X_0, U) \subset \mathcal{E}(Q(t), x_c(t))$

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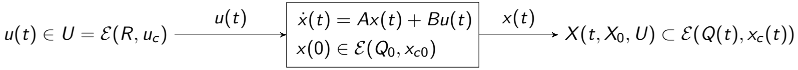


Find the functions $Q : [0, T] \mapsto \mathbb{R}^{n,n}$ and $x_c : [0, T] \mapsto \mathbb{R}^{n,n}$ such that $\forall t > 0, X(t, X_0, U) \subset \mathcal{E}(Q(t), x_c(t))$ [3]

$$\begin{aligned} \dot{Q} &= AQ + QA^\top + hQ + h^{-1}BRB^\top & P(0) &= P_0 \\ \dot{x}_c &= Ax_c + Bu & x_c(0) &= x_{c0} \\ h &= \sqrt{n^{-1} Tr(Q^{-1}BRB^\top)} \end{aligned} \qquad (5)$$

[3] Chernousko F.L. (1999) *What is Ellipsoidal Modelling and How to Use It for Control and State Estimation?* In: Elishakoff I. (eds) Whys and Hows in Uncertainty Modelling. CISM Courses and Lectures, vol 388. Springer, Vienna

# Reachability Analysis for Linear system with Ellipsoids

Ellipsoid of radius $Q \in R^{n,n}$, $Q > 0$ and centered on $x_c \in \mathbb{R}^n$:

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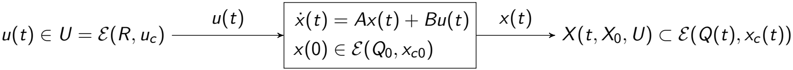


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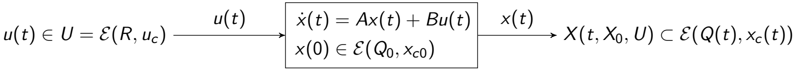
# Reachability Analysis for Linear system with Ellipsoids




$$
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\end{aligned}
\tag{6}
$$

Global optimization problem:

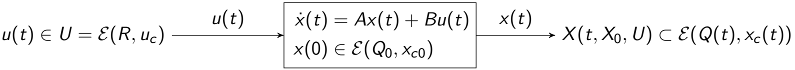
$$
\begin{aligned}
&\text{minimize } Vol\left\{\mathcal{E}\left(Q(T), x_c(T)\right)\right\} \\
&\text{such that Initial Value Problem (6)}
\end{aligned}
$$

- Can be reduced and solved as a Two Boundary Value Problem

⇒ Too complex for simulation!

⇒ Suboptimal solution

# Reachability Analysis for Linear system with Ellipsoids




$$
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# Reachability Analysis for Linear system with Ellipsoids

$u(t) \in U = \mathcal{E}(R, u_c) \xrightarrow{u(t)} \boxed{\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ x(0) \in \mathcal{E}(Q_0, x_{c0}) \end{array}} \xrightarrow{x(t)} X(t, X_0, U) \subset \mathcal{E}(Q(t), x_c(t))$

$$\begin{aligned} \dot{Q} &= AQ + QA^\top + hQ + h^{-1}BRB^\top & P(0) &= P_0 \\ \dot{x}_c &= Ax_c + Bu & x_c(0) &= x_{c0} \\ h &= \sqrt{n^{-1}Tr(Q^{-1}BRB^\top)} \end{aligned} \tag{6}$$

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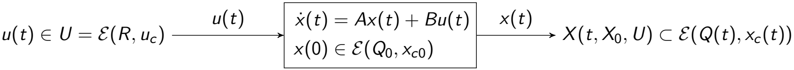
# Reachability Analysis for Linear system with Ellipsoids




$$
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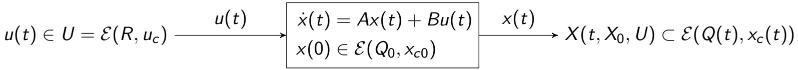


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$$

- Can be reduced and solved as a Two Boundary Value Problem

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# Complete IQC in Temporal Form

Bounded input LTI system:

$$\begin{cases} \dot{x} = Ax + Bu \\ u \in \mathcal{E}(R, u_c) \end{cases}$$

For $w$ a measurable signal, $u$ given:

$$\begin{cases} \dot{x} = Ax + B \begin{bmatrix} u \\ w \end{bmatrix} \\ \dot{x}_q = \begin{bmatrix} x \\ u \\ w \end{bmatrix}^\top M \begin{bmatrix} x \\ u \\ w \end{bmatrix} \\ x_q \geq 0 \end{cases}$$

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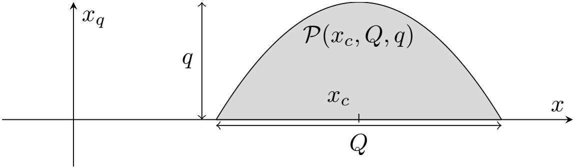
# Reachability Analysis for IQC systems

## Paraboloid Definition

For $x_c \in \mathbb{R}^n$, $Q > 0$ and $q > 0$:

$$\mathcal{P}(x_c, Q, q) = \left\{ \begin{bmatrix} x \\ x_q \end{bmatrix} \in \mathbb{R}^{n+1} \middle| \begin{array}{l} x_q >= 0, \\ (x - x_c)^\top Q^{-1}(x - x_c) + 2\dfrac{x_q}{q} \leq 1 \end{array} \right\} \quad (7)$$



# Reachability Analysis for IQC systems

## Paraboloid Dynamic Equations

$$\begin{cases} \dot{x}_c = & Ax_c + q^\top Q p_x M_{sc} \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right] - B M_w^\top \begin{bmatrix} M_{xw} \\ M_{uw} \end{bmatrix}^\top \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right] + B_u u \\ \dot{q} = & \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right]^\top M_{sc} \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right] \\ \dot{Q} = & \mathcal{H}\left\{AQ\right\} + q^\top Q M_x Q + q^\top \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right]^\top M_{sc} \left[\begin{smallmatrix} x_c \\ u \end{smallmatrix}\right] Q \\ & -q\left(B^\top + q^\top M_{xw}^\top Q\right)^\top M_w^\top \left(B^\top + q^\top M_{xw}^\top Q\right) \end{cases} \tag{8}$$

where

$$M = \begin{bmatrix} M_x & M_{xu} & M_{xw} \\ * & M_u & M_{uw} \\ * & * & M_w \end{bmatrix} \text{ and } \begin{cases} x_c(0) & = x_{c0} \\ q(0) & = q_0 \\ Q(0) & = Q_0 \end{cases},$$

$M_{sc}$ is the Schur complement of $M_w$ of the matrix $M$, i.e.:

$$M_{sc} = \begin{bmatrix} M_x & M_{xu} \\ * & M_u \end{bmatrix} - \begin{bmatrix} M_{xw} \\ M_{uw} \end{bmatrix} M_w^\top \left[\begin{smallmatrix} M_{xw}^\top & M_{uw}^\top \end{smallmatrix}\right].$$

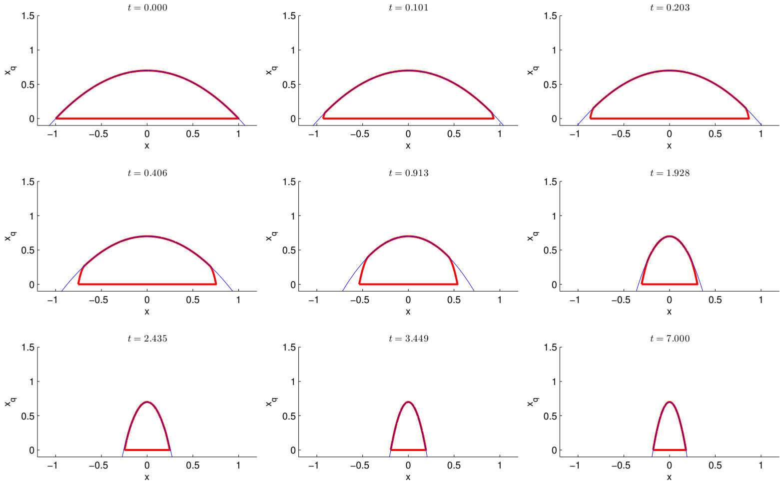
# Reachability Analysis for IQC systems

## Simple System

$$\dot{x} = -x + 0.3w$$
$$\dot{x}_q = x^2 - w^2$$
$$x(0) \in [-1, 1]$$
$$x_q(0) = [0, q_0]$$

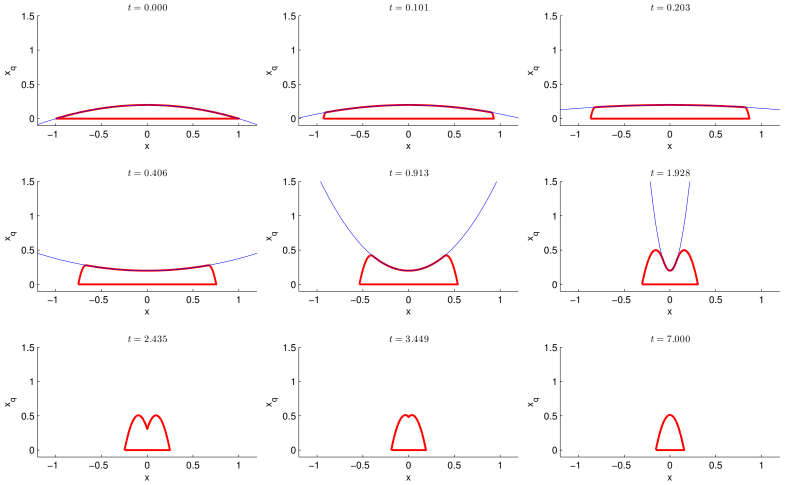
# Reachability Analysis for IQC systems

## Simple System



# Reachability Analysis for IQC systems

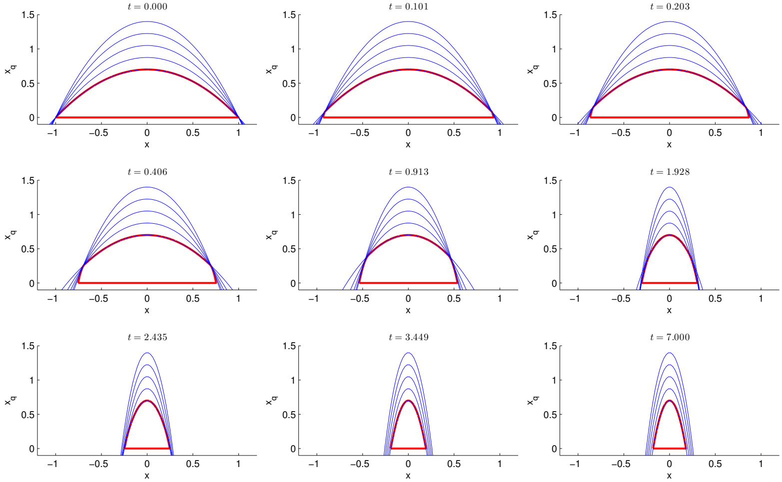
## Simple System




$\Rightarrow$ The paraboloid is not stable!

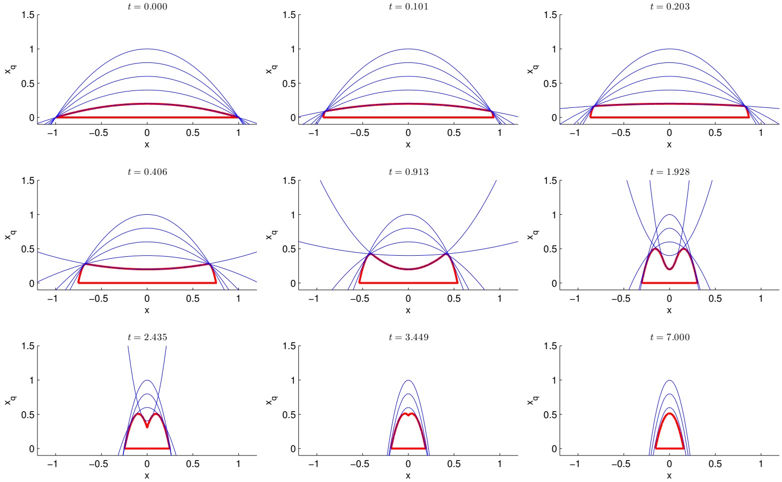
# Reachability Analysis for IQC systems

## Simple System



# Reachability Analysis for IQC systems

## Simple System



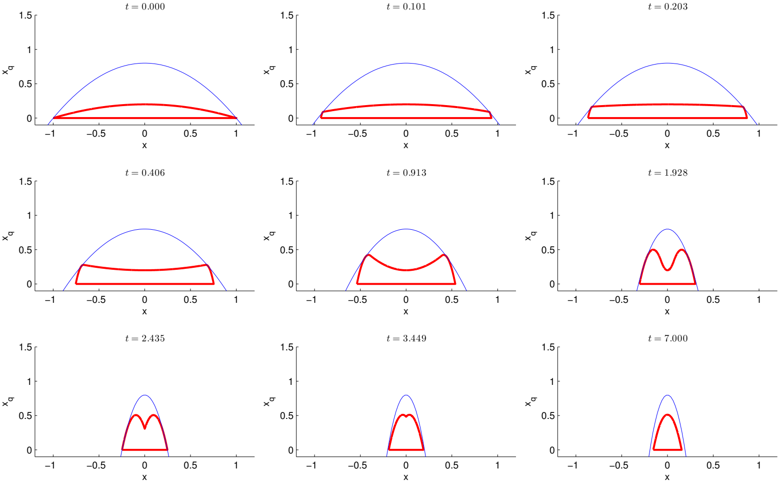
# Controlling the stability

$$\dot{Q}(t) = \cdots + u_Q(t)$$
$$\dot{q}(t) = \cdots + u_q(t)$$

To ensure the overapproximation property:

$$\begin{cases} u_q(t) \geq 0 \\ u_Q(t) \geq 0 \quad \text{(i.e. } u_Q \text{ is SDP)} \end{cases} \tag{9}$$

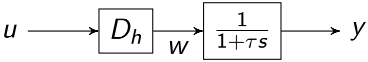
$\Rightarrow$ gives good hope to ensure the stability of the ODE

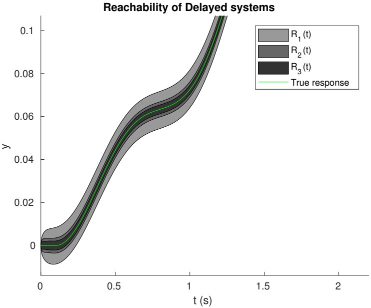


$t = 0.000$
$t = 0.101$
$t = 0.203$
$t = 0.406$
$t = 0.913$
$t = 1.928$
$t = 2.435$
$t = 3.449$
$t = 7.000$
$x_q$
x
1.5
1
0.5
0
−1
−0.5
0
0.5
1

# Delayed System Simulation

Let the following delayed system [4]




$$u(t) = \begin{cases} 0 & \text{if } t < 0 \text{ or } t > 5 \\ 0.3(1-\frac{t}{5})t + 0.1 sin(2\pi t) & \text{otherwise} \end{cases}$$

[4] Seuret, Alexandre, and Frédéric Gouaisbaut. *Hierarchy of LMI conditions for the stability analysis of time-delay systems*. Systems & Control Letters 81 (2015): 1-7.

# Outline

# Current Problems of this Approach & Conclusion

Current problems of this approach:

- The ODE is not stable in general, we have controls on the paraboloids dynamic but no stability theorem!
- We cannot evaluate how conservative we are. Adding paraboloids is possible, adding paraboloids that improve the reachable state computation is difficult (QP problem) and not in the spirit of online reachable set overapproximation.
- IQC are not contractive inequalities: the maximum energy of the reachable set will always increase!

Conclusion:

- use of IQC for verification purposes
- way of simulating CT/DT without running the DT part
- using a family of functions that are adapted to possible geometric shapes of the reachable set