Linear dynamical model approximation

... and its applications

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Feanicses Workshop

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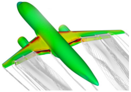
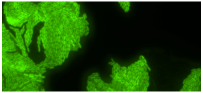
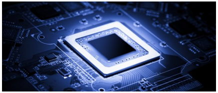
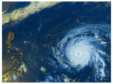
Introduction and motivating examples

Problem statement

Digitalization and computer-based modeling and studies are crucial steps for any system, concept or physical phenomena understanding.



Dynamical models play a pivotal role at many steps of the engineer's work:

- system's understanding through simulation
- system's improvement through optimisation
- system's restitution through measurement and tests
- ...

Introduction and motivating examples

Problem and proposed solution

Problem: numerical dynamical models are too complex and parameter dependent

Finite machine precision, computational burden and memory management:

- induces important time consumption
- generate inaccurate results

Actual numerical tools

- limit the use of class and complexity models

Solution: provide robust and efficient numerical tools to simplify dynamical models

The main objectives are to **save time** and **improve quality**, by

(T) **Time**: speeding up simulation time and reducing computation burden

(Q) **Quality**: enhancing simulation accuracy and memory management

and **extend scope**, by

(S) **Scope**: tailoring larger / more complex dynamical model class to standard tools

Introduction and motivating examples

Scope and considered mathematical dynamical models

Provided **realisation** or **transfer function**

$$\mathcal{S} : (E, A, B, C, D) \text{ or } \mathbf{H}(s)$$

obtained from

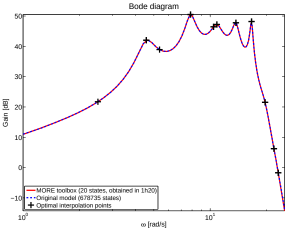
- spatial meshing of PDE
- analytical resolution

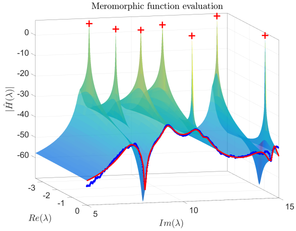
Provided **complex-domain data**

$$\{\imath\omega_i, \mathbf{\Phi}_i\} \text{ or } \{s_i, \mathbf{H}(s_i)\}$$

obtained from

- experiments
- numerical simulation





Introduction and motivating examples

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- experiments
- numerical simulation

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \in \mathbb{R}^n \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \in \mathbb{R}^{n_y}, \\ &\vdots \quad \text{(or an other realization structure)} \\ \mathbf{y}(s) &= H(s)\mathbf{u}(s) \in \mathbb{C}^{n_y} \end{aligned}$$

$$\begin{aligned} \mathbf{\Phi}_i &= \frac{\mathbf{y}(\imath\omega_i)}{\mathbf{u}(\imath\omega_i)} \in \mathbb{C}^{n_y \times n_u}, \\ \mathbf{H}(s_i) &= \frac{\mathbf{y}(s_i)}{\mathbf{u}(s_i)} \in \mathbb{C}^{n_y \times n_u} \end{aligned}$$

Introduction and motivating examples

Scope and considered mathematical dynamical models

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obtained from

- experiments
- numerical simulation

Model approximation paradigm seeks for an **approximation** $\hat{\mathbf{H}}$ (and $\hat{\mathcal{S}}$) which:

- is uniformly "close", *i.e.* given $\mathbf{u}$, $(\mathbf{H} - \hat{\mathbf{H}})\mathbf{u}$ (or $(\mathbf{H}(s_i) - \hat{\mathbf{H}}(s_i))\mathbf{u}$) is "small" in an appropriate sense,
- preserves properties, *e.g.* stability, passivity, subsystem interconnectivity *etc.*
- while procedure is numerically robust and stable, and is simple.

Introduction and motivating examples

Scope and considered mathematical dynamical models [1] [2]

Provided realisation or transfer function

$$\mathcal{S} : (E, A, B, C, D) \text{ or } \mathbf{H}(s)$$

obtained from

- spatial meshing of PDE
- analytical resolution

Provided complex-domain data

$$\{\imath\omega_i, \mathbf{\Phi}_i\} \text{ or } \{s_i, \mathbf{H}(s_i)\}$$

obtained from

- experiments
- numerical simulation

#1 $\mathcal{H}_2$ and $\mathcal{H}_{2,\Omega}$-optimal — $\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$

#2 Infinite dimensional $\mathcal{H}_2$-optimal — $\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$

#3 Delay structured $\mathcal{H}_2$-optimal — $\hat{\mathbf{H}}(s) = \hat{\mathbf{\Delta}}_o(s)\hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}\hat{\mathbf{\Delta}}_i(s)$,

#4 Data-driven interpolation — $\hat{\mathbf{H}}(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B}$

#5 **TDS** stability chart estimation — $\Lambda(\mathbf{H}(s)) \approx \Lambda(\hat{\mathbf{H}}(s))$,

---

[1] P. Vuillemin, *"Frequency-limited model approximation of large-scale dynamical models"*, Ph.D. Onera, ISAE, Toulouse University, Toulouse, France, November 2014.

[2] I. Pontes Duff, *"Large-scale and infinite dimensional dynamical model approximation"*, Ph.D. Onera, ISAE, Toulouse University, Toulouse, France, January 2017.

Introduction and motivating examples

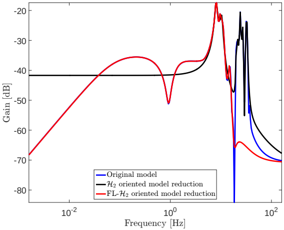
Some applications - #1 business jet aircraft [3]

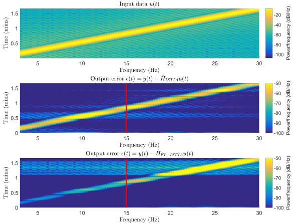
Provided **realisation** or **transfer function**

$$\mathcal{S} : (E, A, B, C, D) \text{ or } \mathbf{H}(s)$$

obtain $\hat{\mathbf{H}}(s)$:

- **ODE** $n = 650$ to $r = 16$
- Frequency-limited $\mathcal{H}_2$ approx.





[3] P. Vuillemin, F. Demourant, J-M. Biannic and C. P-V, *"Stability analysis of a set of uncertain large-scale dynamical models with saturations: application to an aircraft system"*, in IEEE transactions on Control Systems Technology.

Introduction and motivating examples

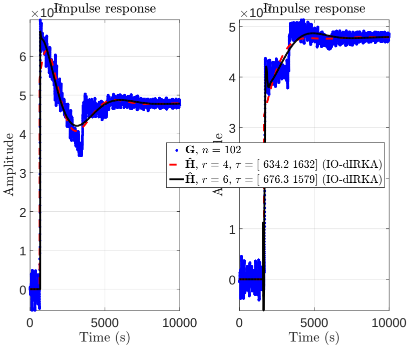
Some applications - #2 Rhin river model [4]

Provided **realisation** or **transfer function**

$$\mathcal{S} : (E, A, B, C, D) \text{ or } \mathbf{H}(s)$$

obtain $\hat{\mathbf{H}}_d(s, \tau)$:

- **PDE** $n = \infty$ to $r = \{4, 6\}$
- $\mathcal{H}_2$ **delayed** model



$$G_e(s,x) = \frac{\lambda_1(s)e^{\lambda_2(s)L+\lambda_1(s)x} - \lambda_2(s)e^{\lambda_1(s)L+\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})}$$

$$G_s(s,x) = \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})}$$

[4] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters.

Introduction and motivating examples

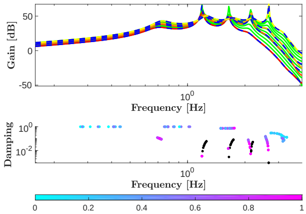
Some applications - #3 flow modeling (N&S equations) [5]

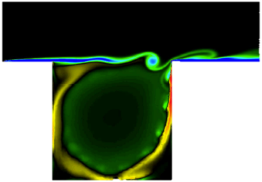
Provided **realisation** or **transfer function**

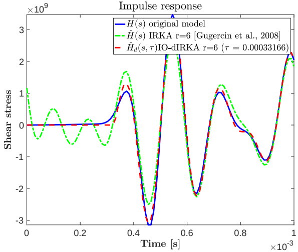
$$\mathcal{S} : (E, A, B, C, D) \text{ or } \mathbf{H}(s)$$

obtain $\hat{\mathbf{H}}(s, p)$ or $\hat{\mathbf{H}}_d(s, p)$:

- **DAE** $n = 650,000$ to $r = 18$
- **Parametric, delayed** $\mathcal{H}_2$ approx.







5 C. P-V and D. Sipp, *"Parametric reduced order dynamical model construction of a fluid flow control problem"*, IFAC LPVS, Grenoble, France, 2015.

Introduction and motivating examples

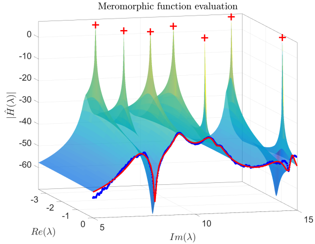
Some applications - #4 ground vibration test [6]

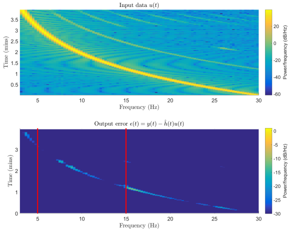
Provided **frequency domain data**

$$\{\imath\omega_i, \mathbf{\Phi}_i\} \text{ or } \{s_i, \mathbf{H}(s_i)\}$$

obtain $\hat{\mathbf{H}}(s)$ or $\hat{\mathbf{H}}(s)$:

- **GVT** models ($i = 1, \ldots, N \approx 1000$)
- Data-driven **meromorphic** approx.





6 C. Meyer, J. Prodigue, G. Broux, O. Cantinaud and C. P-V, *"Ground test for vibration control demonstrator"*, MOVIC'16, Southampton, United Kingdom, 2016.

Introduction and motivating examples

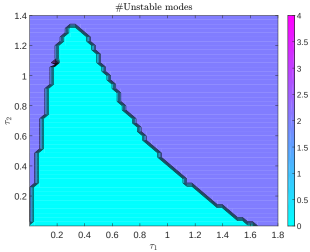
Some applications - #5 high speed network [7]

Provided **realisation** or **transfer function**

$$\mathcal{S} : (E, A, B, C, D, \tau) \text{ or } \mathbf{H}(s, \tau)$$

with delays τ, obtain:

- Approximate functions
- The stability chart



Congestion high speed network system

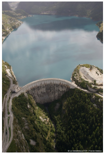
$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau_1) + A_2\mathbf{x}(t-\tau_1-\tau_2)$$

with $\tau_1, \tau_2 \in [0, 1.5]$s

[7] C. P-V, C. Seren, P. Vuillemin, A. Seuret, ..., *"Paper I should I've written"*, in some Journal.

Introduction and motivating examples

Today's talk



Results on finite order model approximation

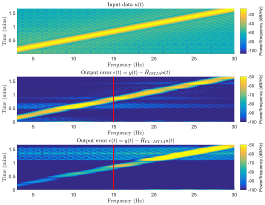
Part 1 over frequency-limited range

Part 2 using input/output delay structured models

- and its application...

Part 1 ... in the aeronautics domain

Part 2 ... and in the hydro-electrical modeling and analysis



Team work

- P. Vuillemin [Onera]
- I. Pontes-Duff [Max Plank Institute]
- C. Seren [Onera]

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Introduction and motivating examples

Problem formulation and settings

Let us consider $\mathbf{H}$, a n_u inputs, n_y outputs linear dynamical system described by the **complex-valued function from $\mathbf{u}$ to $\mathbf{y}$, of order n** (n large or ∞)

$$\mathbf{H} : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}, \tag{1}$$

the model approximation problem consists in finding $\hat{\mathbf{H}}$ of order $r \ll n$

$$\hat{\mathbf{H}} : \mathbb{C} \rightarrow \mathbb{C}^{n_y \times n_u}, \tag{2}$$

that well reproduces the input-output behaviour of $\mathbf{H}$.

Introduction and motivating examples

Problem formulation and settings

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$$\hat{\mathbf{H}} : \mathbb{C} \to \mathbb{C}^{n_y \times n_u}, \quad (2)$$

that well reproduces the input-output behaviour of $\mathbf{H}$. and equipped with a given realization, e.g.

$$\hat{\mathcal{S}} : \begin{cases} \hat{E}\dot{\hat{\mathbf{x}}}(t) &= \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= \hat{C}\hat{\mathbf{x}}(t) \end{cases} \text{ or } \hat{\mathcal{S}}_d : \begin{cases} \hat{E}\dot{\hat{\mathbf{x}}}(t) &= \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\hat{\boldsymbol{\Delta}}_i(\mathbf{u}(t)) \\ \hat{\mathbf{y}}(t) &= \hat{\boldsymbol{\Delta}}_o(\hat{C}\hat{\mathbf{x}}(t)) \end{cases} \quad (3)$$

Introduction and motivating examples

Problem formulation and settings

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$$\hat{\mathcal{S}} : \left\{ \begin{array}{rcl} \hat{E}\dot{\hat{\mathbf{x}}}(t) & = & \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) & = & \hat{C}\hat{\mathbf{x}}(t) \end{array} \right. \quad \text{or } \hat{\mathcal{S}}_d : \left\{ \begin{array}{rcl} \hat{E}\dot{\hat{\mathbf{x}}}(t) & = & \hat{A}\hat{\mathbf{x}}(t) + \hat{B}\hat{\mathbf{\Delta}}_i(\mathbf{u}(t)) \\ \hat{\mathbf{y}}(t) & = & \hat{\mathbf{\Delta}}_o(\hat{C}\hat{\mathbf{x}}(t)) \end{array} \right. \tag{3}$$

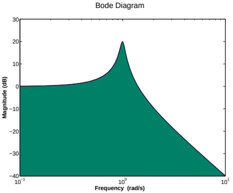
"Well reproduce..."?
$\hat{\mathbf{H}}$ is a "good" approximation of $\mathbf{H}$ if
for the same driving $\mathbf{u}(t)$, $\mathcal{E}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t)$ is "small"

Introduction and motivating examples

Problem formulation and settings[8] [9]

$\mathcal{H}_2$ model approximation

$$\hat{\mathbf{H}} := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_2 \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2} \quad (4)$$



Energy to an impulse input

$$||\mathbf{H}||^2_{\mathcal{H}_2} \quad := \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{tr}\left(\overline{\mathbf{H}(\imath\nu)}\mathbf{H}^T(\imath\nu)\right)d\nu$$

Note that: $||\mathbf{y}(t) - \hat{\mathbf{y}}(t)||_{L_\infty} \leq ||\mathbf{H} - \hat{\mathbf{H}}||_{\mathcal{H}_2}||\mathbf{u}(t)||_{L_2}$

8 S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

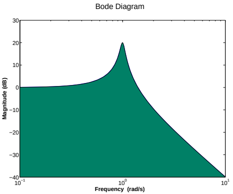
9 K. A. Gallivan, A. Vanderope, and P. Van-Dooren, *"Model reduction of MIMO systems via tangential interpolation"*, SIAM Journal of Matrix Analysis and Application, vol. 26(2), February 2004, pp. 328-349.

Introduction and motivating examples

Problem formulation and settings [8]

Input / output delays structured $\mathcal{H}_2$ model approximation

$$\hat{\mathbf{H}}_d := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_\infty \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2} \tag{4}$$



Energy to an impulse input

$$||\mathbf{H}||^2_{\mathcal{H}_2} \quad := \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{tr}\left(\overline{\mathbf{H}(\imath\nu)}\mathbf{H}^T(\imath\nu)\right) d\nu$$

Note that: $||\mathbf{y}(t) - \hat{\mathbf{y}}(t)||_{L_\infty} \leq ||\mathbf{H} - \hat{\mathbf{H}}||_{\mathcal{H}_2} ||\mathbf{u}(t)||_{L_2}$

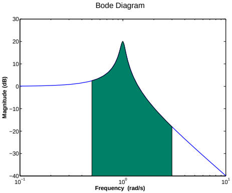
[8] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters.

Introduction and motivating examples

Problem formulation and settings[9] [10]

$\mathcal{H}_{2,\Omega}$ model approximation

$$\hat{\mathbf{H}} := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_\infty \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_{2,\Omega}} \quad (5)$$



Energy (over a finite frequency) to an impulse input

$$||\mathbf{H}||^2_{\mathcal{H}_{2,\Omega}} := \frac{1}{\pi} \int_{\Omega} \mathbf{tr}\left(\overline{\mathbf{H}(\imath\nu)}\mathbf{H}^T(\imath\nu)\right) d\nu$$

9 P. Vuillemin, C. P-V and D. Alazard, *"A Spectral Expression for the Frequency-Limited $\mathcal{H}_2$-norm"*, Available as http://arxiv.org/abs/1211.1858, 2012.

10 P. Vuillemin, C. P-V and D. Alazard, *"Spectral expression for the Frequency-Limited $\mathcal{H}_2$-norm of LTI Dynamical Systems with High Order Poles"*, European Control Conference, 2014, pp. 55-60.

Finite order frequency-limited model approximation

Context and problem description

DASSAULT
AVIATION

Business jet aircraft

- Load aspects (related to weight)
- Vibrations aspects (related to comfort)

Challenges

- Handle flexible models
- Limited frequency range

Finite order frequency-limited model approximation

Petrov-Galerkin approximation

- The state vector trajectories

$$\mathbf{x}(t) = \hat{\mathbf{x}}_1(t)\mathbf{v}_1 + \hat{\mathbf{x}}_2(t)\mathbf{v}_2 + \dots \quad (6)$$

- By setting $\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$ and $\mathbf{span}\,(V) = \mathcal{V}$, the dynamical model becomes,

$$\hat{\mathcal{S}} : \begin{cases} EV\dot{\hat{\mathbf{x}}}(t) &= AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + r(t) \\ \hat{\mathbf{y}}(t) &= CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{cases} \quad (7)$$

  The residual $r(t) \in \mathbb{R}^n$ accounts for the fact that $V\hat{\mathbf{x}}(t)$ will not be an exact solution to the dynamical equation.

- The residual $r(t)$ is then constrained to be orthogonal to a subspace $W \in \mathbb{R}^{n\times r}$, where $\mathbf{span}\,(W) = \mathcal{W}$, i.e.:

$$W^T r(t) = 0 \quad (8)$$

A projection method consists then in seeking for an approximation $\hat{\mathbf{x}}(t)$ of $\mathbf{x}(t)$, by imposing the following two conditions:

$$\hat{\mathbf{x}}(t) \in \mathcal{V} \text{ and } \left(EV\dot{\hat{\mathbf{x}}}(t) - \left(AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\right)\right) \perp \mathcal{W} \quad (9)$$

Finite order frequency-limited model approximation

Petrov-Galerkin approximation

- The state vector trajectories

$$\mathbf{x}(t) = \hat{\mathbf{x}}_1(t)\mathbf{v}_1 + \hat{\mathbf{x}}_2(t)\mathbf{v}_2 + \dots \tag{6}$$

- By setting $\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$ and $\mathbf{span}\,(V) = \mathcal{V}$, the dynamical model becomes,

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  The residual $r(t) \in \mathbb{R}^n$ accounts for the fact that $V\hat{\mathbf{x}}(t)$ will not be an exact solution to the dynamical equation.

- The residual $r(t)$ is then constrained to be orthogonal to a subspace $W \in \mathbb{R}^{n\times r}$, where $\mathbf{span}\,(W) = \mathcal{W}$, i.e.:

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A projection method consists then in seeking for an approximation $\hat{\mathbf{x}}(t)$ of $\mathbf{x}(t)$, by imposing the following two conditions:

$$\hat{\mathbf{x}}(t) \in \mathcal{V} \text{ and } \left(EV\dot{\hat{\mathbf{x}}}(t) - \left(AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\right)\right) \perp \mathcal{W} \tag{9}$$

Finite order frequency-limited model approximation

Petrov-Galerkin approximation

- The state vector trajectories

$$\mathbf{x}(t) = \hat{\mathbf{x}}_1(t)\mathbf{v}_1 + \hat{\mathbf{x}}_2(t)\mathbf{v}_2 + \dots \tag{6}$$

- By setting $\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t)$ and $\mathbf{span}\,(V) = \mathcal{V}$, the dynamical model becomes,

$$\hat{\mathcal{S}} : \left\{ \begin{array}{rcl} EV\dot{\mathbf{x}}(t) & = & AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + r(t) \\ \hat{\mathbf{y}}(t) & = & CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{array} \right. \tag{7}$$

  The residual $r(t) \in \mathbb{R}^n$ accounts for the fact that $V\hat{\mathbf{x}}(t)$ will not be an exact solution to the dynamical equation.

- The residual $r(t)$ is then constrained to be orthogonal to a subspace $W \in \mathbb{R}^{n\times r}$, where $\mathbf{span}\,(W) = \mathcal{W}$, i.e.:

$$W^T r(t) = 0 \tag{8}$$

A projection method consists then in seeking for an approximation $\hat{\mathbf{x}}(t)$ of $\mathbf{x}(t)$, by imposing the following two conditions:

$$\hat{\mathbf{x}}(t) \in \mathcal{V} \text{ and } \left(EV\dot{\hat{\mathbf{x}}}(t) - \left(AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\right)\right) \perp \mathcal{W} \tag{9}$$

Finite order frequency-limited model approximation

Petrov-Galerkin approximation

By setting

$$\hat{\mathbf{x}}(t) \in \mathcal{V} \text{ and } EV\dot{\hat{\mathbf{x}}}(t) - \Big(AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\Big) \perp \mathcal{W} \tag{10}$$

or equivalently

$$\hat{\mathbf{x}}(t) \in \mathcal{V} \text{ and } W^T\left(EV\dot{\hat{\mathbf{x}}}(t) - \Big(AV\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\Big)\right) = 0 \tag{11}$$

One then obtains,

$$\hat{\mathcal{S}} : \begin{cases} W^T EV\dot{\hat{\mathbf{x}}}(t) &= W^T AV\hat{\mathbf{x}}(t) + W^T B\mathbf{u}(t) + 0 \\ \hat{\mathbf{y}}(t) &= CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{cases} \tag{12}$$

Moreover, the approximated full state vector can be reconstructed if needed as,

$$\mathbf{x}(t) \approx V\hat{\mathbf{x}}(t) \tag{13}$$

This is known as the **Petrov-Galerkin projection** framework

Finite order frequency-limited model approximation

Approximation by projection

Comments about V and W

Let us consider the (oblique) projection,

$$\hat{\mathcal{S}} : \begin{cases} W^T E V \dot{\hat{\mathbf{x}}}(t) &= W^T A V \hat{\mathbf{x}}(t) + W^T B \mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= C V \hat{\mathbf{x}}(t) + D \mathbf{u}(t) \end{cases} \quad (14)$$

$$\hat{\mathbf{x}}_0 = W^T \mathbf{x}_0 \in \mathbb{R}^r \quad (15)$$

Lemma

Choosing two different bases V' and W' that respectively span the same subspaces $\mathcal{V}$ and $\mathcal{W}$ result in the same reconstructed solution $\mathbf{x}(t)$.

Thus, subspaces are relevant, not basis

Finite order frequency-limited model approximation

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Finite order frequency-limited model approximation

Approximation by projection

- A reduced order model is uniquely defined by its projector $\Pi_{V,W} = VW^T$
- The projector $\Pi_{V,W}$ is itself uniquely defined by the two subspaces

$$\begin{array}{c} \mathbf{span}\,(V) = \mathcal{V} \\ \mathbf{span}\,(W) = \mathcal{W} \end{array} \quad (16)$$

- $\mathcal{V}$ and $\mathcal{W}$ belong to the Grassmann manifold $\mathcal{G}(r, n)$: known as the set of all subspaces of dimension r in $\mathbb{R}^n$

Reduced Order Model $\leftrightarrow (\mathcal{V}, \mathcal{W})$
How to find V and W (criterion)?

Finite order frequency-limited model approximation

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Reduced Order Model $\leftrightarrow (\mathcal{V},\mathcal{W})$
How to find V and W (criterion)?

Finite order frequency-limited model approximation

Standard methods

Truncation (mostly dense)

- Modal, $\{V, W\}$ are eigenvectors subspaces
- Balanced, $\{V, W\}$ come from Lyapunov and SVD subspaces
- Singular perturbation, $\{V, W\}$ come from Lyapunov and SVD subspaces
- ...

Interpolation (mostly sparse)

- √ Moment matching (quite general formulation)
- √ Rational (Padé, Markov, generalized), $\{V, W\}$ are Krylov subspaces
- √ Multi-point ($\mathcal{H}_2$ optimal or not), $\{V, W\}$ are generalized Krylov subspaces

Hybrid (mostly dense)

- √ Balanced / multi-point, $\{V, W\}$ are generalized Krylov and SVD subspaces

Finite order frequency-limited model approximation

Moment matching problem

Moment matching problem

Given a **LTI** model, $\mathbf{H}$ can be expanded at $\sigma \in \mathbb{C}$ as

$$\mathbf{H}(s)\big|_{\sigma} = \sum_{i=0}^{\infty} \eta_i(\sigma)(s-\sigma)^i \tag{17}$$

The problem consists in finding a reduced-order model $\hat{\mathbf{H}}$ with

$$\hat{\mathbf{H}}(s)\Big|_{\sigma} = \sum_{i=0}^{\infty} \hat{\eta}_i(\sigma)(s-\sigma)^i, \tag{18}$$

such that,

$$\eta_i(\sigma) = \hat{\eta}_i(\sigma) \quad \forall i \in 1, \dots, r. \tag{19}$$

Numerically ill-conditioned to explicitly matching them
Use Krylov subspaces

Finite order frequency-limited model approximation

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Finite order frequency-limited model approximation

Implicit moment matching and Krylov subspace

Definition: Krylov subspace $\mathcal{K}_r$

Given $A \in \mathbb{R}^{n\times n}$ and $\mathbf{v} \in \mathbb{R}^n$, the r-th order Krylov subspace, denoted $\mathcal{K}_r(A, \mathbf{v})$ is defined as

$$\mathcal{K}_r(A, \mathbf{v}) := \textbf{span}\left(\mathbf{v}, A\mathbf{v}, \ldots, A^{r-1}\mathbf{v}\right) \quad (20)$$

Krylov subspaces are "everywhere" in linear algebra:

- solution of linear equations $A\mathbf{x} = \mathbf{b}$,
- eigenvalue computation,
- approximate solutions of Lyapunov equations,
- and model reduction...

For moment matching, we are interested in :

- $\mathcal{K}_r(A, B)$: to match at $\sigma = \infty$,
- $\mathcal{K}_r\left(A^{-1}, B\right)$: to match at $\sigma = 0$,
- $\mathcal{K}_r\left((\sigma I_n - A)^{-1}, B\right)$: for matching at $\sigma \in \mathbb{C}$,
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Finite order frequency-limited model approximation

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Finite order frequency-limited model approximation

Implicit moment matching and Krylov subspace

Reminder: Petrov-Galerkin (oblique) projection
Let V, $W \in \mathbb{R}^{n\times r}$ be such that $W^T V = I_r$,

$$\left\{ \begin{array}{rcl} E\dot{\mathbf{x}}(t) & = & A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) & = & C\mathbf{x}(t) + D\mathbf{u}(t) \end{array} \right. \Rightarrow \left\{ \begin{array}{rcl} W^T E V \dot{\hat{\mathbf{x}}}(t) & = & W^T A V \hat{\mathbf{x}}(t) + W^T B\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) & = & CV\hat{\mathbf{x}}(t) + D\mathbf{u}(t) \end{array} \right. \quad (21)$$

Theorem: Two-sided moment matching

Let us consider a n-th order **SISO LTI** dynamical model $\mathcal{S} : (A, B, C, D, E)$ and $\sigma \in \mathbb{C}$ s.t. $\sigma E - A$ is full rank. If $V, W \in \mathbb{C}^{n\times r}$ are full column rank matrices s.t.

$$\begin{array}{rcccl} \mathcal{K}_r\left((\sigma E - A)^{-1}, (\sigma E - A)^{-1} B\right) & \subseteq & \mathcal{V} & = & \mathbf{span}\,(V) \\ \mathcal{K}_r\left((\sigma E - A)^{-T}, (\sigma E - A)^{-T} C^T\right) & \subseteq & \mathcal{W} & = & \mathbf{span}\,(W) \end{array} \quad (22)$$

then, the $2r$ first moments of the reduced-order model $\hat{\mathbf{H}}$, obtained by projection, matches the $2r$ first moments of $\mathbf{H}$ at σ, i.e.

$$\eta_i(\sigma) = \hat{\eta}_i(\sigma), \quad i = 1, \dots, 2r \quad (23)$$

Finite order frequency-limited model approximation

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Finite order frequency-limited model approximation

One-sided Krylov algorithm in ∞, 0 and σ

Require: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $r \in \mathbb{N}$
1: Construct **span**$(V) = \mathcal{K}_r(A, B)$
2: Apply projectors V and $W = V$ {bi-orthogonality}
Ensure: $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$

Require: $A^{-1} \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $r \in \mathbb{N}$
1: Construct **span**$(V) = \mathcal{K}_r(A^{-1}, B)$
2: Apply projectors V and $W = V$ {bi-orthogonality}
Ensure: $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$

Require: $(\sigma I_n - A)^{-1} \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $r \in \mathbb{N}$
1: Construct **span**$(V) = \mathcal{K}_r\left((\sigma I_n - A)^{-1}, B\right)$
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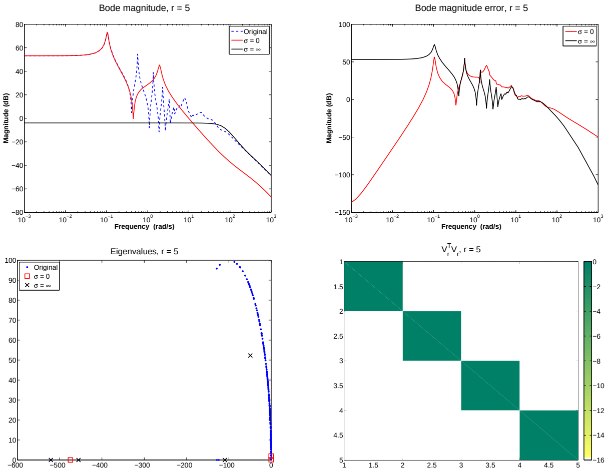
One-sided Krylov algorithm in ∞, 0 and σ

One-sided Krylov algorithm in ∞, 0 and σ



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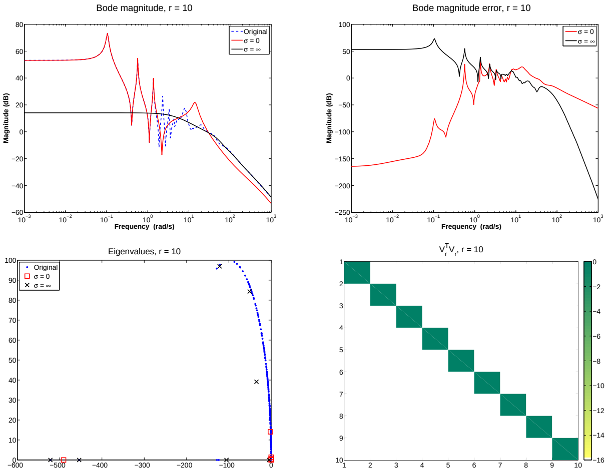

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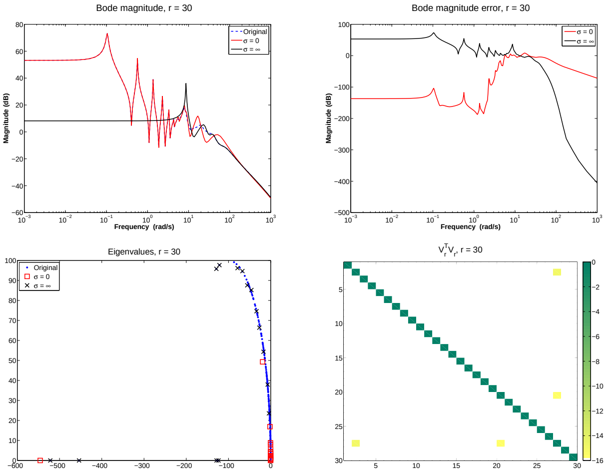

Finite order frequency-limited model approximation

Two-sided Krylov algorithm

Algorithm: Two-sided Krylov Algorithm (KA2)

Require: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, $r \in \mathbb{N}$, $\sigma \in \mathbb{C}$

1: Construct $\textbf{span}(V) = \mathcal{K}_r\Big((\sigma I_n - A)^{-1}, B\Big)$

2: Construct $\textbf{span}(W) = \mathcal{K}_r\Big((\sigma I_n - A)^{-T}, C^T\Big)$

3: Set $W \leftarrow W(W^TV)^{-T}$ {to ensure $W^TV = I_r$}

4: Apply projectors V and W

Ensure: $V, W \in \mathbb{C}^{n \times r}$ and $W^TV = I_r$

11 Y. Saad, *"Iterative methods for sparse linear systems"*, SIAM, 2003.

Finite order frequency-limited model approximation

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3: Set $W \leftarrow W(W^T V)^{-T}$ {to ensure $W^T V = I_r$}

4: Apply projectors V and W

Ensure: $V, W \in \mathbb{C}^{n \times r}$ and $W^T V = I_r$

Matches twice more moments thus enhancing the approximation

Instead of 2 Arnoldi procedures, on can use the Lanczos Algorithm[11],

- it directly builds V and W with $W^T V = I_r$,
- it is numerically cheaper,
- but breakdowns can occur.

---

[11] Y. Saad, *"Iterative methods for sparse linear systems"*, SIAM, 2003.

Finite order frequency-limited model approximation

Generalized Krylov and multi-point moment matching

By considering the union of several Krylov subspaces, *i.e.*

Generalized Krylov subspaces.

Theorem: Two-sided moment matching at several points

Let us consider a n-th order SISO LTI dynamical model $\mathcal{S} : (A, B, C, D, E)$ and $\{\sigma_1, \dots, \sigma_{n_\sigma}\} \in \mathbb{C}^{n_\sigma}$ s.t. $\forall i$, $(\sigma_i E - A)$ is full rank. If $V, W \in \mathbb{C}^{n \times r}$ are full column rank matrices s.t.

$$\begin{array}{rcccl} \bigcup_{k=1}^{n_\sigma} \mathcal{K}_{r_k}\left((\sigma_k E - A)^{-1}, (\sigma_k E - A)^{-1}B\right) & \subseteq & \mathcal{V} & = & \mathbf{span}(V) \\ \bigcup_{k=1}^{n_\sigma} \mathcal{K}_{r_k}\left((\sigma_k E - A)^{-T}, (\sigma_k E - A)^{-T}C^T\right) & \subseteq & \mathcal{W} & = & \mathbf{span}(W) \end{array} \quad (24)$$

then, the $2r_k$ first moments of the reduced-order model $\hat{\mathbf{H}}$, obtained by projection, matches the $2r_k$ first moments of $\mathbf{H}$ at each σ_k, i.e. for $k = 1, \dots, n_\sigma$,

$$\eta_i(\sigma_k) = \hat{\eta}_i(\sigma_k), \quad i = 0, \dots, 2r_k - 1. \quad (25)$$

Finite order frequency-limited model approximation

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Finite order frequency-limited model approximation

Generalized Krylov and multi-point moment matching

Theorem: First-order optimality conditions for the $\mathcal{H}_2$ problem

Let $\hat{\mathbf{H}}$ be a r-th order asymptotically stable model with semi-simple poles only. If $\hat{\mathbf{H}}$ is solution of the $\mathcal{H}_2$ approximation problem, then

$$\begin{array}{rcl} \hat{\mathbf{c}}_i^T \mathbf{H}(-\hat{\lambda}_i) & = & \hat{\mathbf{c}}_i^T \hat{\mathbf{H}}(-\hat{\lambda}_i) \\ \mathbf{H}(-\hat{\lambda}_i)\hat{\mathbf{b}}_i & = & \hat{\mathbf{H}}(-\hat{\lambda}_i)\hat{\mathbf{b}}_i \\ \hat{\mathbf{c}}_i^T \mathbf{H}'(-\hat{\lambda}_i)\hat{\mathbf{b}}_i & = & \hat{\mathbf{c}}_i^T \hat{\mathbf{H}}'(-\hat{\lambda}_i)\hat{\mathbf{b}}_i \end{array} \quad (26)$$

where $\hat{\lambda}_i$ and $\{\hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\}$ are the poles and associated residues of $\hat{\mathbf{H}}(s)$.

[12] P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21, no. 12, pp. 53-62, December 2008.

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where $\hat{\lambda}_i$ and $\{\hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\}$ are the poles and associated residues of $\hat{\mathbf{H}}(s)$.

- the reduced-order model is a bi-tangential Hermite interpolant of the large-scale model at the opposite of its poles,
- these conditions can be obtained from the state-space formulation (see [12])

[12] P. Van-Dooren, K. A. Gallivan, and P. A. Absil, *"$\mathcal{H}_2$-optimal model reduction of MIMO systems"*, Applied Mathematics Letters, vol. 21, no. 12, pp. 53-62, December 2008.

Finite order frequency-limited model approximation

Generalized Krylov and multi-point moment matching

- The optimality conditions can be viewed as a set of coupled equations,

$$\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right) = F_2\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right) \quad \text{and} \quad \left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right) = G_2\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right). \tag{27}$$

which admit a fixed point at every stationary point of $\mathcal{J}$.
$\hookrightarrow$this suggests an iterative procedure

$$\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right)_{k+1} = F_2\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right)_{k+1}, \quad \left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right)_{k+1} = G_2\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right)_k \tag{28}$$

Interpolatory approach (MIMO IRKA or ITIA):

- initially proposed for SISO models as *Iterative Rational Krylov Algorithm* in [13],
- the step $\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right)_{k+1} = F_2\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right)_{k+1}$ is done by solving a small-scale eigenvalue problem,
  $\hookrightarrow$assumes that $\hat{A}$ is diagonalisable
- the step $\left(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}\right)_{k+1} = G_2\left(\hat{\lambda}_i, \hat{\mathbf{c}}_i, \hat{\mathbf{b}}_i\right)_k$ is done by tangential interpolation through Krylov subspaces (projection).

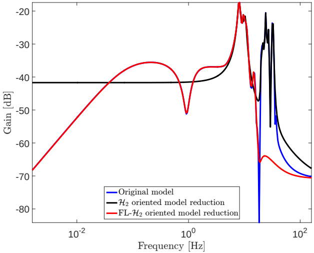
[13] S. Gugercin, A.C. Antoulas and C. Beattie, *"A rational Krylov iteration for optimal $\mathcal{H}_2$ model reduction"*, Proceedings of the International Symposium on Mathematical Theory of Networks and Systems, 2006.

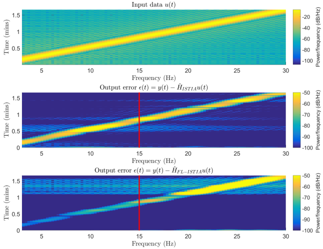
Finite order frequency-limited model approximation

And now a frequency-limited version[14]

Mixing SVD and interpolatory methods

- V is constructed by Lyapunov and SVD... and especially, frequency-limited gramians
- W is constructed by interpolatory method
- follow the iterative scheme





14 P. Vuillemin et al., *"Paper on going :)"*, eventually in a Journal.

Finite order delay structured model approximation

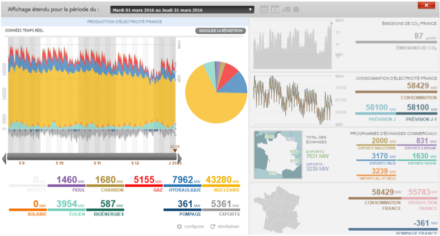
Context and problem description[15]

Hydraulics green electricity ($\approx 10\%$)

- Dams
- Run-of-the-river

Run-of-the-river ($\approx 5\%$)

- In France, provides 3.6GW
- Rely open-channel hydraulic systems
- Need for analysis and control



[15] http://alsace.edf.com/actions/fonctionnement-des-centrales-hydroelectriques-sur-le-rhin/

Finite order delay structured model approximation

Context and problem description

Modelling assumptions

- No discharge, no infiltration, one dimensional flow, small bed slope, small stream line, negligible vertical acceleration
- Input $\mathbf{u}$: boundary conditions $q_e(t)$ and $q_s(t)$
- Output $\mathbf{y}$: water depth
- t, x are the time and spatial variables

Uniform cross section

From equations

$$\mathbf{H}(s,x) \in \mathbb{C}^{1\times 2} \tag{29}$$

an irrational transfer function at a given position $x = x_m$.

Non uniform cross section

From a dedicated software

$$\{\omega_i, \Phi_i(x)\} \in (\mathbb{C}, \mathbb{C}^{1\times 2}) \tag{30}$$

an input/output transfer data collection at a given position $x = x_m$.
(not in this presentation)

Finite order delay structured model approximation

Context and problem description[16]

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\ \frac{\partial Q}{\partial t} + \frac{\partial (Q^2/S)}{\partial x} + gS\frac{\partial H}{\partial x} &= gS(I-J), \end{aligned} \tag{31}$$

$x \in [0\ ;\ L]$ is the spatial variable, $H(x,t)$ the water depth, $S(x,t)$ the wetted area, $Q(x,t)$ the discharge...

16 V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th European Control Conference (ECC'16), Aalborg, Denmark, July, 2016.

Finite order delay structured model approximation

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$x \in [0 \; ; \; L]$ is the spatial variable, $H(x,t)$ the water depth, $S(x,t)$ the wetted area, $Q(x,t)$ the discharge...

1. Apply linearisation at $\delta = (H_0, Q_0)$, which are both x_m dependent
2. Apply Laplace around equilibrium
3. Find solutions of $h(s, x_m)$, $q(s, x_m)$ & identify coefficient (boundary conditions)
4. Full order Loewner interpolation of the filtered function
5. Approximation with and without delay
6. Back to original problem

[16] V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th European Control Conference (ECC'16), Aalborg, Denmark, July, 2016.

Finite order delay structured model approximation

Context and problem description

$$h(s, x_m, \delta) = \mathbf{G}_e(s, x_m, \delta) q_e(s) - \mathbf{G}_s(s, x_m, \delta) q_s(s) \quad (32)$$

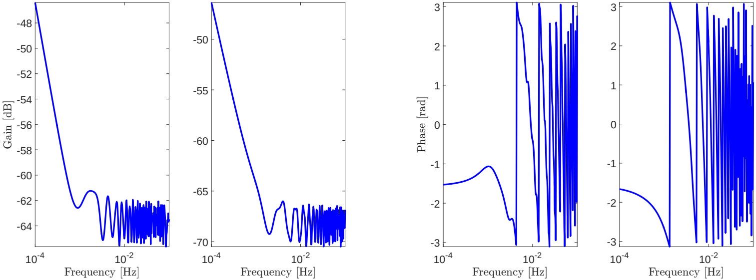
$$\begin{aligned} \mathbf{G}_e(s, x_m, \delta) &= \frac{\lambda_1(s) e^{\lambda_2(s)L + \lambda_1(s)x_m} - \lambda_2(s) e^{\lambda_1(s)L + \lambda_2(s)x_m}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \\ \mathbf{G}_s(s, x_m, \delta) &= \frac{\lambda_1(s) e^{\lambda_1(s)x_m} - \lambda_2(s) e^{\lambda_2(s)x_m}}{B_0 s(e^{\lambda_1(s)L} - e^{\lambda_2(s)L})} \end{aligned} \quad (33)$$

- Irrational transfer function
- Infinite order equation

Finite order delay structured model approximation

Context and problem description

$$h(s, x_m, \delta) = \mathbf{G}_e(s, x_m, \delta) q_e(s) - \mathbf{G}_s(s, x_m, \delta) q_s(s) \tag{32}$$



- Delay behaviour is obvious
- Not $\mathcal{H}_2$ function

Finite order delay structured model approximation

Delay structured ROM

Model-based (uniform)

From equations

$$\mathbf{H}(s, x_m) \in \mathbb{C}^{1\times 2} \quad (33)$$

an irrational transfer function.

Data-based (non uniform)

From a dedicated software

$$\{\omega_i, \Phi_i(x_m)\} \in (\mathbb{C}, \mathbb{C}^{1\times 2}) \quad (34)$$

an input/output transfer data collection.

... the objective is to approximate it by

$$\begin{array}{rcl} \dot{\hat{\mathbf{x}}}(t) & = & \hat{A}(\delta)\hat{\mathbf{x}}(t) + \hat{B}(\delta)\mathbf{u}(t-\tau(\delta)) \\ \hat{\mathbf{y}}(t) & = & \hat{C}(\delta)\hat{\mathbf{x}}(t) + \hat{D}(\delta)\mathbf{u}(t-\tau(\delta)), \end{array} \quad (35)$$

- $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times n_u}$, $\hat{C} \in \mathbb{R}^{n_y\times r}$ and $\hat{D} \in \mathbb{R}^{n_y\times n_u}$
- which are linearly dependent on δ,
- and $\tau(\delta) \in \mathbb{R}_+^{n_u}$ is an input delay vector.

Finite order delay structured model approximation

Delay structured ROM

Model-based (uniform)

From equations

$$\mathbf{H}(s, x_m) \in \mathbb{C}^{1\times 2} \quad (33)$$

an irrational transfer function.

Data-based (non uniform)

From a dedicated software

$$\{\omega_i, \Phi_i(x_m)\} \in (\mathbb{C}, \mathbb{C}^{1\times 2}) \quad (34)$$

an input/output transfer data collection.

... from the open-channel example

$$h(s, x, \delta) = \mathbf{G}_e(s, x, \delta) q_e(s) - \mathbf{G}_s(s, x, \delta) q_s(s) \quad (35)$$

one seeks the input delayed r-th order rational function

$$\begin{array}{rcl} \hat{h}(s, \delta) & = & \hat{\mathbf{G}}_\mathbf{e}(s, \delta) q_e(s) - \hat{\mathbf{G}}_\mathbf{s}(s, \delta) q_s(s) \\ \hat{\mathbf{G}}_\mathbf{e}(x_m, s, \delta) & = & \mathbf{R}_\mathbf{e}(s, \delta) e^{-\tau_e(\delta) s} \\ \hat{\mathbf{G}}_\mathbf{s}(x_m, s, \delta) & = & \mathbf{R}_\mathbf{s}(s, \delta) e^{-\tau_s(\delta) s} \end{array} \quad (36)$$

Finite order delay structured model approximation

Delay structured ROM - an approach when the delay is known

If delays are a-priori known functions, approximation can be done on the shifted function

$$\tilde{h}(s, x_m, \delta) = \mathbf{G}_e(s, x_m, \delta)e^{+\tau_e(\delta)s}q_e(s) - \mathbf{G}_s(s, x_m, \delta)e^{+\tau_s(\delta)s}q_s(s) \quad (37)$$

- then apply Loewner
- and go back to $h(s, x, \delta)$

or

- apply **TF-IRKA**[17]
- and go back to $h(s, x, \delta)$

The Loewner approach is preferred for practical reasons in[18].

However, is the fixed delays the best idea? What if you don't a priori know them?

[17] C.A. Beattie, and S. Gugercin, *"Realization-independent $\mathcal{H}_2$-approximation"*, in ProceedingsProceedings of the 51st IEEE CDC, USA, December, 2012.

[18] V. Dalmas, G. Robert, C. Poussot-Vassal, I. Pontes Duff and C. Seren, *"From infinite dimensional modelling to parametric reduced order approximation: Application to open-channel flow for hydroelectricity"*, in Proceedings of the 15th ECC, Denmark, July, 2016.

Finite order delay structured model approximation

Delay structured ROM - the I/O delay structured alternative [19]

Input / output delays structured $\mathcal{H}_2$ model approximation

$$\hat{\mathbf{H}}_d := \arg \min_{\substack{\mathbf{G} \in \mathcal{H}_\infty \\ \mathbf{rank}(\mathbf{G}) = r \ll n}} ||\mathbf{H} - \mathbf{G}||_{\mathcal{H}_2} \quad (38)$$

where $\hat{\mathbf{H}}_d = \hat{\mathbf{\Delta}}_o \hat{\mathbf{H}} \hat{\mathbf{\Delta}}_i$.

$\mathcal{H}_2$ interpolatory conditions in the delay free case no longer apply

- due to the exponential terms in the transfer function...
- ...which implies a non symmetrical inner product (next slide)
- dedicated conditions need to be derived

[19] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters.

Finite order delay structured model approximation

Delay structured ROM - $\mathcal{H}_2$-inner product issue

$$\mathbf{G}(s) = \frac{1}{s+1} = \frac{\psi}{s-\mu},\ \mathbf{H}(s) = \frac{1}{s+2} = \frac{\hat{\phi}}{s-\hat{\lambda}},\ (\mu = -1 \text{ and } \hat{\lambda} = -2)$$

- Delay-free case[20] (Lemma 2.4):

$$\langle \mathbf{G}, \mathbf{H} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu) = \psi \frac{\hat{\phi}}{-\mu - \hat{\lambda}} = \frac{1}{3} = \hat{\phi} \frac{\psi}{-\hat{\lambda} - \mu} = \hat{\phi} \mathbf{G}(-\hat{\lambda}) = \langle \mathbf{H}, \mathbf{G} \rangle_{\mathcal{H}_2}$$

  $\mathcal{H}_2$-inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

- Delay dependent case[21] :

[20] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

[21] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters.

Finite order delay structured model approximation

Delay structured ROM - $\mathcal{H}_2$-inner product issue

$$\mathbf{G}(s) = \frac{1}{s+1} = \frac{\psi}{s-\mu}, \ \mathbf{H}(s) = \frac{1}{s+2} = \frac{\hat{\phi}}{s-\hat{\lambda}}, \ (\mu = -1 \text{ and } \hat{\lambda} = -2)$$

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$\mathcal{H}_2$-inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

- Delay dependent case[21] :
  Let $\mathbf{H}(s) \leftarrow \mathbf{H}(s)e^{-s}$.
  Let move the delay as $\langle \mathbf{H}e^{-s}, \mathbf{G} \rangle_{\mathcal{H}_2} = \langle \mathbf{H}, \mathbf{G}e^{s} \rangle_{\mathcal{H}_2}$, one obtains :

$$\langle \mathbf{G}, \mathbf{H}e^{-s} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu)e^{\mu} = \frac{1}{3}e^{-1} \neq \frac{1}{3}e^{2} = \hat{\phi} \mathbf{G}(-\hat{\lambda})e^{-\hat{\lambda}} = \langle \mathbf{H}, \mathbf{G}e^{s} \rangle_{\mathcal{H}_2}.$$

Symmetric $\mathcal{H}_2$-inner product does not provide the same result any more.

[20] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

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Finite order delay structured model approximation

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$\mathcal{H}_2$-inner product can be computed using pole-residues decomposition of $\mathbf{G}$ or $\mathbf{H}$.

- Delay dependent case[21] :
  Now using the $\mathcal{H}_2$-inner product between $\mathbf{H}e^{-s}$ and $\mathbf{G}$ using the extended formula:

$$\langle \mathbf{G}, \mathbf{H}e^{-s} \rangle_{\mathcal{H}_2} = \psi \mathbf{H}(-\mu) e^{\mu} = \psi \frac{\hat{\phi}}{-\mu - \hat{\lambda}} e^{\mu} = \frac{1}{3} e^{-1}.$$

  Which modifies the optimality conditions

[20] S. Gugercin and A C. Antoulas and C A. Beattie, *"$\mathcal{H}_2$ Model Reduction for Large Scale Linear Dynamical Systems"*, SIAM Journal on Matrix Analysis and Applications, vol. 30(2), June 2008, pp. 609-638.

[21] I. Pontes Duff, C. P-V and C. Seren, *"$\mathcal{H}_2$-optimal model approximation by input / output-delay structured reduced order models"*, in Systems & Control Letters.

Finite order delay structured model approximation

Delay structured ROM - SISO I/O delay ROM approximation $\mathcal{H}_2$ optimality conditions

$$\mathbf{G}(s) = \sum_{j=1}^{N} \frac{\psi_j}{s-\mu_j} \text{ and } \hat{\mathbf{H}}(s) = \sum_{k=1}^{n} \frac{\hat{\phi}_k}{s-\hat{\lambda}_k}, \text{ and } \hat{\mathbf{H}}_d = \mathbf{H}e^{-\tau s}$$

Theorem: SISO case

Interpolatory conditions on $\tilde{\mathbf{G}}$

$$\hat{\mathbf{H}}(-\hat{\lambda}_k) = \tilde{\mathbf{G}}(-\hat{\lambda}_k),\ \hat{\mathbf{H}}'(-\hat{\lambda}_k) = \tilde{\mathbf{G}}'(-\hat{\lambda}_k), \quad (39)$$

Delay conditions

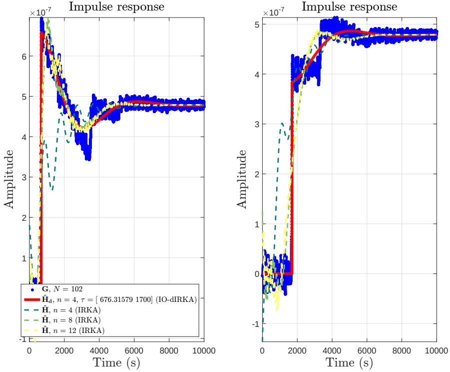
$$\sum_{j=1}^{n} \mu_j \psi_j \left(\sum_{k=1}^{r} \frac{\phi_k}{\mu_j + \hat{\lambda}_k} \right) e^{\tau \mu_j} = 0. \quad (40)$$

for all $k = 1 \dots r, l = 1 \dots n_u$ and $m = 1 \dots n_y$ where $\tilde{\mathbf{G}}(s)$ is given by

$$\tilde{\mathbf{G}}(s) = \sum_{j=1}^{n} \frac{\psi_j}{s-\mu_j} e^{\tau \mu_j} \text{ (pole / residue decomposition needed).} \quad (41)$$

Finite order delay structured model approximation

Delay structured ROM (delay structured $\mathcal{H}_2$, approximate)



Errors

- **IO-dIRKA** (r=4) $\mathbf{4.34672 \times 10^{-15}}$
- **IRKA** (r=4): 6.43008×10^{-15}
- **IRKA** (r=8): $\mathbf{4.06717 \times 10^{-15}}$
- **IRKA** (r=12): $\mathbf{3.76871 \times 10^{-15}}$

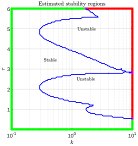
- Rational model obtained by Loewner filtered
- Rational ROM, $r = 4$ filtered (with and without input delay structure)

Epilogue and perspectives

What to keep in mind?

Model approximation...

- plays a pivotal role in many industrial and applicative contexts
- interesting tool for analysis, optimization...



MOR Toolbox...

- is tailored to a large family of representation
- ... from input output data
- ... from irrational transfer functions
- ... from rational functions or a set of ODE
- Has been applied in different industrial contexts

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Linear dynamical model approximation

... and its applications

Charles Poussot-Vassal

May 2018
Feanicses Workshop

http://mordigitalsystems.fr/