Semidefinite Approximations of Reachable Sets for Polynomial Systems

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Joint work with

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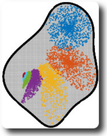

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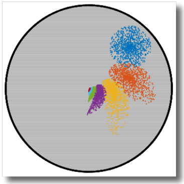
The RS Problem in Discrete Time



Initial conditions $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) \geqslant 0\} \quad h_j \in \mathbb{R}[\mathbf{x}]$

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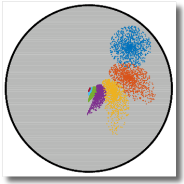

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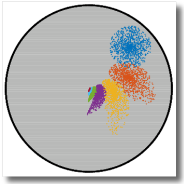

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Reachable Set (RS) of admissible trajectories
$\mathbf{X}^\infty := \{(\mathbf{x}_t)_{t \in \mathbb{N}} : \mathbf{x}_{t+1} = f(\mathbf{x}_t), \forall t \in \mathbb{N}, \mathbf{x}_0 \in \mathbf{X}_0\}$

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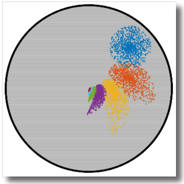

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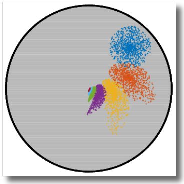
$\mathbf{X}^\infty = \bigcup_{t\in\mathbb{N}} f^t(\mathbf{X}_0) \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)

Tractable approximations of RS $\mathbf{X}^\infty$?

The RS Problem in Continuous Time



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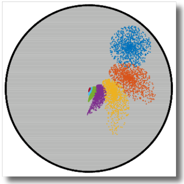

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$\mathbf{X}^T := \{(\mathbf{x}(t)) : \dot{\mathbf{x}} = f(\mathbf{x}), \forall t \in [0, T], \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{X}_0\}$

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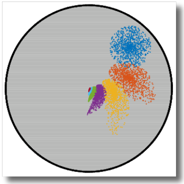

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$\mathbf{X}^\infty := \lim_{t \to \infty} \mathbf{X}^T \subseteq \mathbf{X} \subset \mathbb{R}^n$ (box or ball)

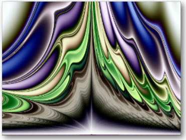
Tractable approximations of RS $\mathbf{X}^\infty$?

Motivations

- Occurs in several contexts :
  1. program analysis: fixpoint computation

```
toyprogram ($x_1$,$x_2$)
    requires ($0.25 \leqslant x_1 \leqslant 0.75$ && $0.25 \leqslant x_2 \leqslant 0.75$)
        ;
    while ($x_1^2 + x_2^2 \leqslant 1$){
        $x_1 = x_1 + 2x_1x_2$;
          $x_2 = 0.5(x_2 - 2x_1^3)$;
     }
```

  2. hybrid systems, biology: Neuron Model, Growth Model



  3. control: integrator, Hénon map

Related work: RS

1. Contractive methods based on LP relaxations and polyhedra projection [Bertsekas 72]

2. Extension to nonlinear systems [Harwood et al. 16]

3. Bernstein/Krivine-Handelman representations [Ben Sassi- et al. 15, Ben Sassi et al. 12]

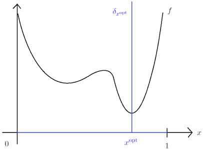
$\oplus$ LP relaxations $\implies$ scalability

$\ominus$ Convex approximations of nonconvex sets $\implies$ coarse

$\ominus$ No convergence guarantees (very often)

Related work: Lasserre hierarchy

Cast a polynomial optimization problem as an *infinite-dimensional* LP over measures [Lasserre 2001]



$$f^\star := \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f(\mathbf{x}) d\mu$$

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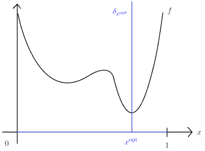
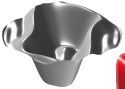
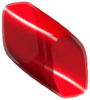


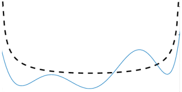




⇝ Regions of attraction [Henrion-Korda 14]

⇝ Maximum invariants [Korda et al. 13]

⇝ Invariant 1D densities [Henrion 2012]



⇝ Maximal positively invariants [Oustry-Tacchi-Henrion 2019]

Related work: Lasserre hierarchy

5 SDP approximation of polynomial images of semialgebraic sets [M.-Henrion-Lasserre 15]

- $\mathbf{X}_1 := f(\mathbf{X}_0) \subseteq \mathbf{X}$, with $\mathbf{X} \subset \mathbb{R}^n$ a box or a ball
  $\implies$ Discrete-time system with a single iteration

- Approximation of image measure supports
  $\implies$ certified SDP over approximations of $\mathbf{X}_1$

- $\mathbf{X}_t := f^t(\mathbf{X}_0)$

  ⊖ $\deg f^t = d \times t \implies$ very expensive computation

  ⊖ Would only approximate $\mathbf{X}_t$ and not $\mathbf{X}^\infty$

Contributions

- General framework to approximate $\mathbf{X}^\infty$

  $\oplus$ **No discretization** is required

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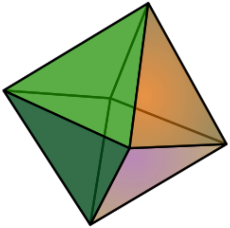
- Finite-dimensional SDP relaxations

- $\mathbf{X}^\infty \subseteq \mathbf{X}_r^\infty = \{\mathbf{x} \in \mathbf{X} : w_r(\mathbf{x}) \geqslant 1\}$
  $\oplus$ Strong convergence guarantees
  $\lim_{r\to\infty} \mathrm{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$
  $\oplus$ Compute w_r by solving one **semidefinite program**

What is Semidefinite Programming?

- Linear Programming (LP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \mathbf{A}\,\mathbf{z} \geqslant \mathbf{d}\;.$$



- Linear cost $\mathbf{c}$
- Linear inequalities "$\sum_i A_{ij} z_j \geqslant d_i$"

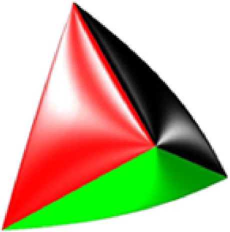
Polyhedron

What is Semidefinite Programming?

- Semidefinite Programming (SDP):

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \ .$$

  - Linear cost $\mathbf{c}$
  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
  - Linear matrix inequalities "$\mathbf{F} \succcurlyeq 0$" ($\mathbf{F}$ has nonnegative eigenvalues)



Spectrahedron

What is Semidefinite Programming?

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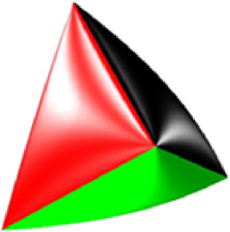

Spectrahedron

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  - Symmetric matrices $\mathbf{F}_0, \mathbf{F}_i$
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SDP for Polynomial Optimization

- Prove **polynomial inequalities** with SDP:

$$f(a,b) := a^2 - 2ab + b^2 \geqslant 0 \ .$$

- Find $\mathbf{z}$ s.t. $f(a,b) = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix}$.
- Find $\mathbf{z}$ s.t. $a^2 - 2ab + b^2 = z_1 a^2 + 2 z_2 ab + z_3 b^2 \qquad (\mathbf{A}\,\mathbf{z} = \mathbf{d})$
- $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_1} z_1 + \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathbf{F}_2} z_2 + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{F}_3} z_3 \succcurlyeq \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}_0}$

SDP for Polynomial Optimization

- Choose a cost $\mathbf{c}$ e.g. $(1, 0, 1)$ and solve:

$$\min_{\mathbf{z}} \quad \mathbf{c}^\top \mathbf{z}$$
$$\text{s.t.} \quad \sum_i \mathbf{F}_i z_i \succcurlyeq \mathbf{F}_0 \,, \quad \mathbf{A}\,\mathbf{z} = \mathbf{d} \,.$$

- Solution $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succcurlyeq 0$ (eigenvalues 0 and 2)
- $a^2 - 2ab + b^2 = \begin{pmatrix} a & b \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\succcurlyeq 0} \begin{pmatrix} a \\ b \end{pmatrix} = (a - b)^2 \,.$
- Solving **SDP** $\implies$ Finding **SUMS OF SQUARES** certificates

SDP for Polynomial Optimization

NP hard General Problem: $f^* := \min_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x})$

- Semialgebraic set $\mathbf{X} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

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  $\mathbf{X} = [0,1]^2 = \{\mathbf{x} \in \mathbb{R}^2 : x_1(1 - x_1) \geqslant 0, \quad x_2(1 - x_2) \geqslant 0\}$

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$$\overbrace{x_1 x_2}^{f} =$$

$$-\frac{1}{8} + \overbrace{\frac{1}{2}\left(x_1 + x_2 - \frac{1}{2}\right)^2}^{\sigma_0} + \overbrace{\frac{1}{2}}^{\sigma_1} \overbrace{x_1(1 - x_1)}^{g_1} + \overbrace{\frac{1}{2}}^{\sigma_2} \overbrace{x_2(1 - x_2)}^{g_2}$$

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- Sums of squares (SOS) σ_i

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- Sums of squares (SOS) σ_i
- Bounded degree:

  $\mathcal{Q}_r(\mathbf{X}) := \left\{ \sigma_0 + \sum_{j=1}^{l} \sigma_j g_j, \text{ with } \deg \sigma_j \, g_j \leqslant 2r \right\}$

SDP for Polynomial Optimization

- **Hierarchy of SDP relaxations**:
  $m_r := \sup_m \Big\{ m : f - m \in \mathcal{Q}_r(\mathbf{X}) \Big\}$

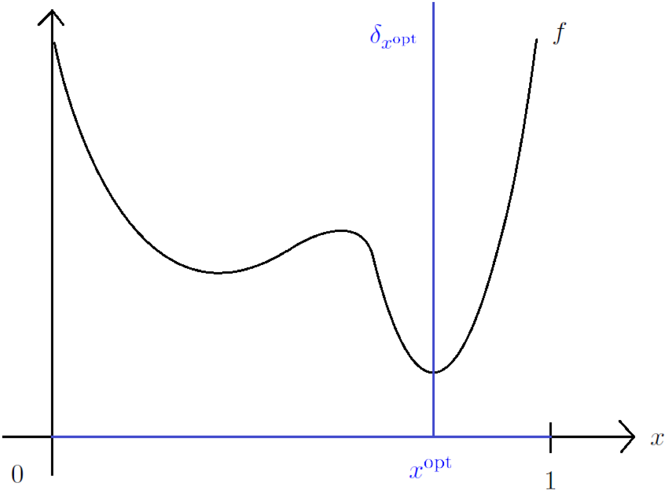
- Convergence guarantees $m_r \uparrow f^*$ [Lasserre 01]

- Can be computed with SDP solvers (CSDP, SDPA)

- **"No Free Lunch" Rule**: $\binom{n+2r}{n}$ SDP variables

Primal-dual Moment-SOS [Lasserre 01]

$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$



Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{X}$ if there exists a finite measure μ supported on $\mathbf{X}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{X}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n .$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{X})$: space of **probability measures** supported on $\mathbf{X}$
- $\mathcal{Q}(\mathbf{X})$: combining **sums of squares** and polynomials g_j from $\mathbf{X}$

Polynomial Optimization Problems (POP)

(Primal) = (Dual)

$$\begin{aligned} \inf \quad & \int_{\mathbf{X}} f \, d\mu \\ \text{s.t.} \quad & \mu \in \mathcal{M}_+(\mathbf{X}) \end{aligned} \quad = \quad \begin{aligned} \sup \quad & m \\ \text{s.t.} \quad & m \in \mathbb{R} \ , \\ & f - m \in \mathcal{Q}(\mathbf{X}) \end{aligned}$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{X})$
- Truncated quadratic module $\mathcal{Q}_r(\mathbf{X})$

Lasserre's Hierarchy for Polynomial Optimization

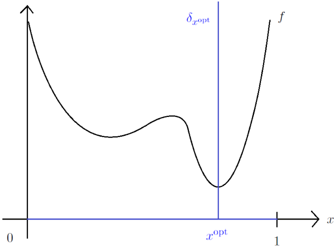
(Moment) (SOS)

$$\begin{aligned} \inf \quad & \sum_\alpha f_\alpha \, \mathbf{z}_\alpha & = \quad & \sup \quad m \\ \text{s.t.} \quad & \mathbf{M}_{r-r_j}(g_j \, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leqslant j \leqslant l, & & \text{s.t.} \quad m \in \mathbb{R} \;, \\ & z_0 = 1 & & f - m \in \mathcal{Q}_r(\mathbf{X}) \end{aligned}$$

Characterizing the RS

CHARACTERIZE A VALUE

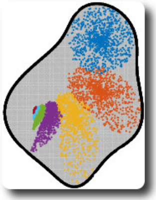
$$f^* = \inf_{\mathbf{x} \in \mathbf{X}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{X})} \int_{\mathbf{X}} f \, d\mu$$



Dirac measure $\mu^\star = \delta_{\mathbf{x}^{\mathrm{opt}}}$

CHARACTERIZE A SET

?

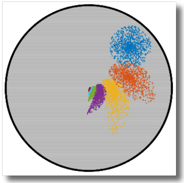


Lebesgue measure $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Occupation Measures and Liouville's Equation

$\mathbf{x}_{t+1} = f(\mathbf{x}_t) \quad \mathbf{x}_0 \in \mathbf{X}_0$

$\mathbf{x}_1 = f(\mathbf{x}_0) \ldots \mathbf{x}_t = f(\mathbf{x}_{t-1})$



- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$
- **Pushforward** $f_\# : \mathcal{M}_+(\mathbf{X}_0) \to \mathcal{M}_+(\mathbf{X})$

$$\mu_1(\mathbf{A}) = f_\# \mu_0(\mathbf{A}) := \mu_0(f^{-1}(\mathbf{A}))$$

- $f_\# \mu_0$ is the **image measure** of μ_0 under f

Occupation Measures and Liouville's Equation

- Let $\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$ and

$$\mu_1 = f_{\#}\, \mu_0$$
$$\dots$$
$$\mu_t = f_{\#}\, \mu_{t-1}$$
$$\nu_t = \sum_{i=0}^{t-1} \mu_i = \sum_{i=0}^{t-1} f_{\#}^i\, \mu_0$$

- The measures μ_t, ν_t, μ_0 satisfy **Liouville's Equation**:

$$\mu_t + \nu_t = f_{\#}\, \nu_t + \mu_0$$

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_{\#}\, \mu_{0,t}$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**

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  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**
- Lebesgue measure $\lambda_{\mathbf{X}^T}$ on $\mathbf{X}^T := \bigcup_{t=0}^{T} \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

Occupation Measures and Liouville's Equation

- Lebesgue measure $\lambda_{\mathbf{X}_t}$ on $\mathbf{X}_t = f^t(\mathbf{X}_0)$
- $\exists\, \mu_{0,t} \in \mathcal{M}_+(\mathbf{X}_0)$ s.t. $\lambda_{\mathbf{X}_t} = f^t_{\#}\, \mu_{0,t}$
  $\implies \lambda_{\mathbf{X}_t}$ satisfies **Liouville's Equation.**
- Lebesgue measure $\lambda_{\mathbf{X}^T}$ on $\mathbf{X}^T := \bigcup_{t=0}^{T} \mathbf{X}_t$
  $\implies \lambda_{\mathbf{X}^T}$ satisfies **Liouville's Equation** by superposition

$$\lambda_{\mathbf{X}^T} + \nu^T = f_{\#}\, \nu^T + \mu_0^T$$

average **occupation measure** ν^T: measures time spent in $\mathbf{X}^T$

Volume Assumption

Discrete Time

Define $\mathbf{Y}^0 := \mathbf{X}^0$ and $\mathbf{Y}^t := \mathbf{X}_t \backslash \mathbf{X}^{t-1}$.

$$\lim_{T \to \infty} \sum_{t=0}^{T} t \, \mathrm{vol} \, \mathbf{Y}^t < \infty.$$

Volume Assumption

Discrete Time

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Volume Assumption

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Proof

- $\lambda_{\mathbf{X}^T} = \sum_{t=0}^{T} \lambda_{\mathbf{Y}^t} \to \lambda_{\mathbf{X}^\infty}$ as $T \to \infty$
- $\mu_t + \nu_t = f_\# \nu_t + \mu_{0,t} \implies \nu^T := \sum_{t=0}^{T} \nu_t$ has mass $\leqslant \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t$

Volume Assumption

Continuous Time

Define $\tau(\mathbf{x})$ = minimal time to reach $\mathbf{x}$.

$$\frac{1}{\mathrm{vol}(\mathbf{X})} \int_{\mathbf{X}^\infty} \tau(\mathbf{x}) d\mathbf{x} < \infty.$$

Volume Assumption

Continuous Time

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Lemma

Under Volume Assumption, $\lambda_{\mathbf{X}^\infty}$ satisfies **Liouville's Equation**

Infinite Primal LP for Discrete RS

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \nu = f_{\#} \nu + \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
& \mu_0 \in \mathcal{M}_+(\mathbf{X}_0) \,, \quad \mu, \nu \in \mathcal{M}_+(\mathbf{X})
\end{aligned}
$$

Infinite Primal LP for Continuous RS

$$
\begin{aligned}
p^T := \sup_{\mu_0, \mu, \nu} \quad & \int_{\mathbf{X}} \mu \\
\text{s.t.} \quad & \int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X} \\
& \mu + \operatorname{div}(f \nu) = \mu_0 \\
& \mu \leqslant \lambda_{\mathbf{X}} \\
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\end{aligned}
$$

$$
\int_{\mathbf{X}} v(\mathbf{x}) \operatorname{div} f \nu = - \int_{\mathbf{X}} \operatorname{grad} v(\mathbf{x}) \cdot f(\mathbf{x}) d\nu
$$

Infinite Primal LP for Discrete/Continuous RS

Lemma

Volume Assumption $\Longrightarrow$ optimal solution $\mu^\star = \lambda_{\mathbf{X}^\infty}$

Primal-dual LP in Discrete Time

Primal LP

$$p^T := \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu$$

s.t.

$$\int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X}$$

$$\mu + \nu = f_{\#} \nu + \mu_0$$

$$\mu \leqslant \lambda_{\mathbf{X}}$$

$$\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$$

$$\mu, \nu \in \mathcal{M}_+(\mathbf{X})$$

Dual LP

$$d^T := \inf_{u,v,w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x}$$

s.t.

$$v \in \mathcal{C}_+(\mathbf{X}_0)$$

$$w - v - 1 \in \mathcal{C}_+(\mathbf{X})$$

$$w \in \mathcal{C}_+(\mathbf{X})$$

$$u + v \circ f - v \in \mathcal{C}_+(\mathbf{X})$$

$$u \geqslant 0$$

$$u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X})$$

Primal-dual LP in Continuous Time

Primal LP

$$p^T := \sup_{\mu_0, \mu, \nu} \int_{\mathbf{X}} \mu$$

s.t. $\int_{\mathbf{X}} \nu \leqslant T \operatorname{vol} \mathbf{X}$

$\mu + \operatorname{div}(f \nu) = \mu_0$

$\mu \leqslant \lambda_{\mathbf{X}}$

$\mu_0 \in \mathcal{M}_+(\mathbf{X}_0)$

$\mu, \nu \in \mathcal{M}_+(\mathbf{X})$

Dual LP

$$d^T := \inf_{u, v, w} \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x}$$

s.t. $v \in \mathcal{C}_+(\mathbf{X}_0)$

$w - v - 1 \in \mathcal{C}_+(\mathbf{X})$

$w \in \mathcal{C}_+(\mathbf{X})$

$u + \operatorname{grad} v \cdot f \in \mathcal{C}_+(\mathbf{X})$

$u \geqslant 0$

$u \in \mathbb{R}, v, w \in \mathcal{C}(\mathbf{X})$

Zero Duality Gap

Lemma

1. $p^T = d^T$ and $\exists$ minimizing sequence (u_k, v_k, w_k) for dual LP.
2. $u_k = 0 \implies$ Volume Assumption $\implies p^T = d^T = \operatorname{vol} \mathbf{X}^\infty$

SDP Strengthening of the Dual LP

Discrete Time

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + T u) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\
& u + v \circ f - v \in \mathcal{Q}_{rd}(\mathbf{X}) \\
& w \in \mathcal{Q}_r(\mathbf{X}) \\
& u \geqslant 0
\end{aligned}
$$

SDP Strengthening of the Dual LP

Continuous Time

$$
\begin{aligned}
d_r^T := \inf_{u,v,w} \quad & \int_{\mathbf{X}} (w(\mathbf{x}) + Tu) d\mathbf{x} \\
\text{s.t.} \quad & v \in \mathcal{Q}_r(\mathbf{X}_0) \\
& w - v - 1 \in \mathcal{Q}_r(\mathbf{X}) \\
& u + \operatorname{grad} v \cdot f \in \mathcal{Q}_{r+d}(\mathbf{X}) \\
& w \in \mathcal{Q}_r(\mathbf{X}) \\
& u \geqslant 0
\end{aligned}
$$

Strong Convergence Properties

Theorem

Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.

Strong Convergence Properties

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Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \setminus \mathbf{X}^\infty$ have nonempty interior.

1. No duality gap between primal and dual SDP: $p_r^T = d_r^T$.
2. Dual SDP has optimal solution (u_r, v_r, w_r):

$$\lim_{r \to \infty} \int_{\mathbf{X}} |w_r + u_r T - \mathbf{1}_{\mathbf{X}^\infty}| \, d\mathbf{x} = 0 \,.$$

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3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.

Strong Convergence Properties

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Assume that $\mathbf{X}^0, \mathbf{X}^\infty, \mathbf{X} \backslash \mathbf{X}^\infty$ have nonempty interior.

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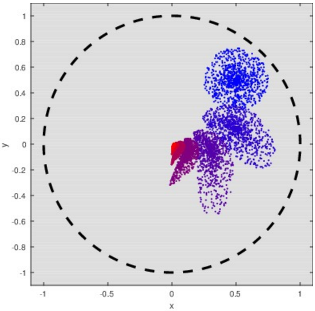
3. Let $\mathbf{X}_r^T := \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supseteq \mathbf{X}^T$.
4. $u_r = 0 \Rightarrow$ Volume Assumption $\Rightarrow \lim_{r \to \infty} \mathrm{vol}(\mathbf{X}_r^\infty \backslash \mathbf{X}^\infty) = 0$.

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

$$x_1^+ := \frac{1}{2}(x_1 + 2x_1x_2),$$
$$x_2^+ := \frac{1}{2}(x_2 - 2x_1^3),$$



$\mathbf{X}_2^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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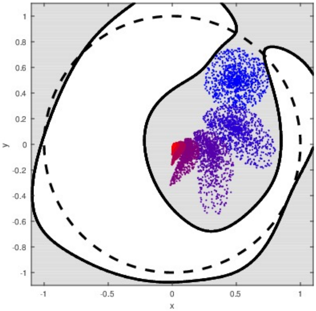

$\mathbf{X}_3^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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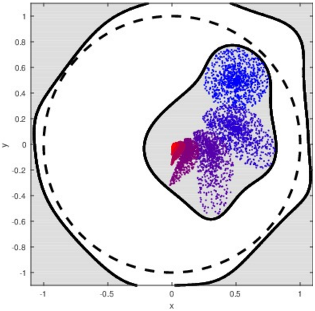

$\mathbf{X}_4^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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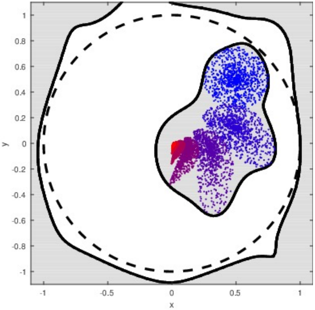

$\mathbf{X}_5^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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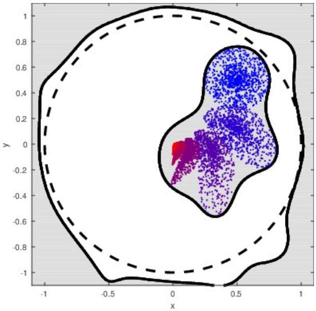

$\mathbf{X}_6^\infty$

Toy Example

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : (x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 \leqslant \frac{1}{4}\}$ under

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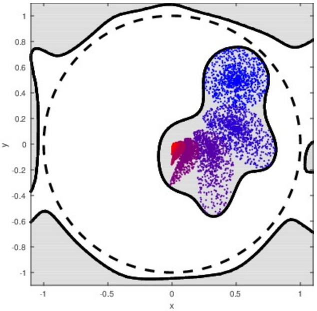
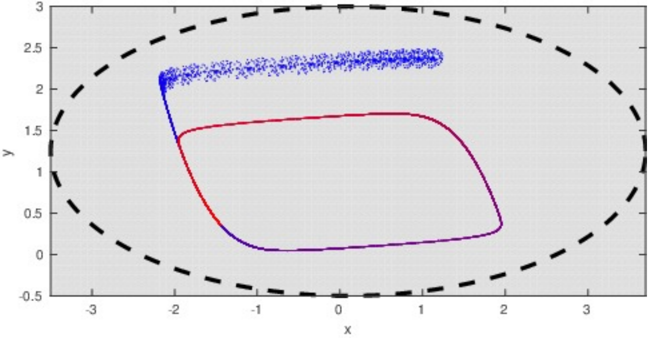


$\mathbf{X}_7^\infty$

FitzHugh-Nagumo Neuron Model

Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

$$x_1^+ := x_1 + 0.2(x_1 - x_1^3/3 - x_2 + 0.875)\,,$$
$$x_2^+ := x_2 + 0.2(0.08(x_1 + 0.7 - 0.8x_2))\,,$$



$\mathbf{X}_2^\infty$

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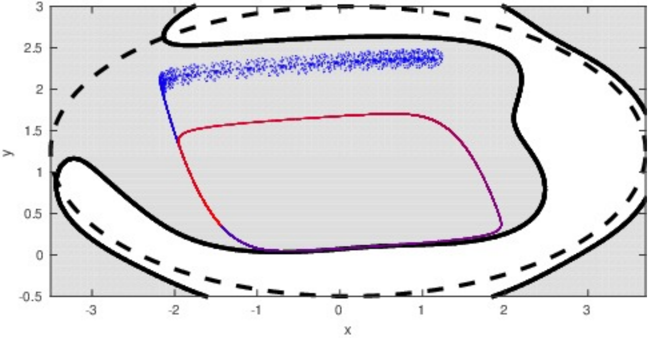

$\mathbf{X}_3^\infty$

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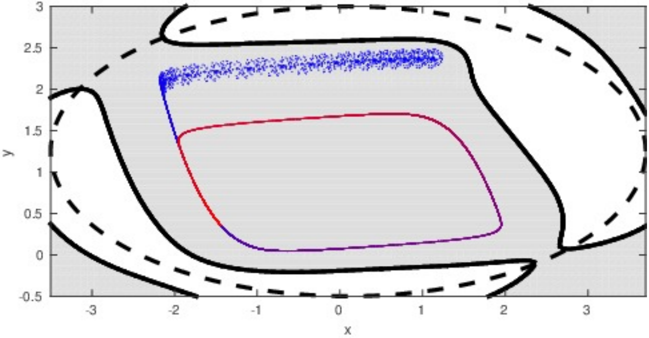

$\mathbf{X}_4^\infty$

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Trajectories from $\mathbf{X}_0 := [1, 1.25] \times [2.25, 2.5]$ under

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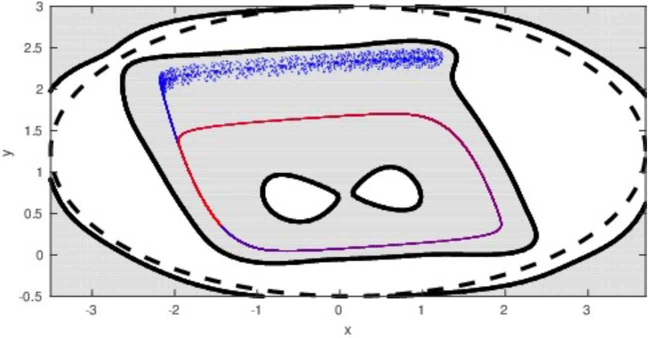

$\mathbf{X}_5^\infty$

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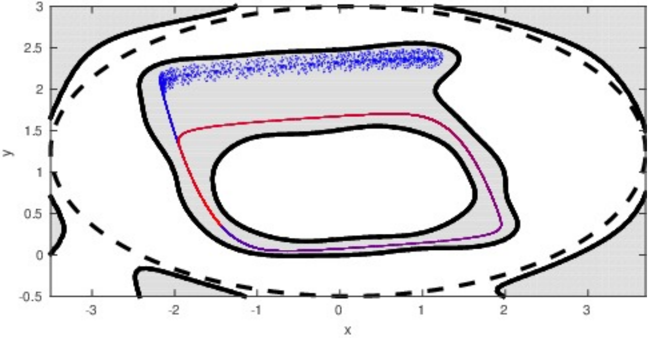

$\mathbf{X}_6^\infty$

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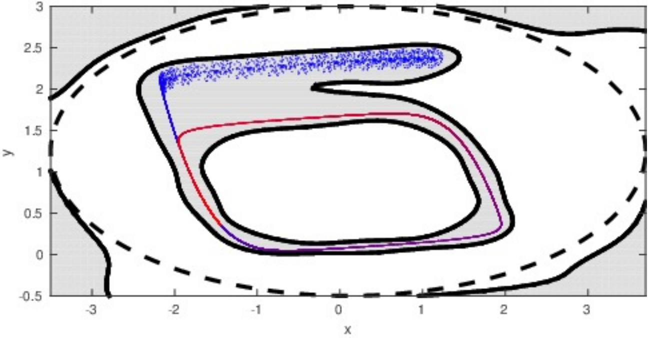

$\mathbf{X}_7^\infty$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

$$x_1^+ := x_1^2 - x_2^2 + c_1 \,,$$
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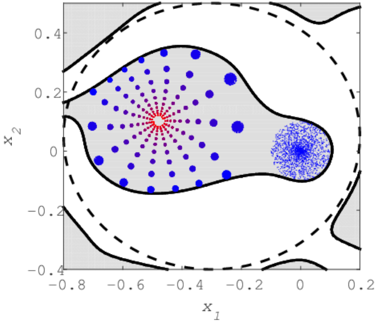

$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = 0.2$

Julia Map

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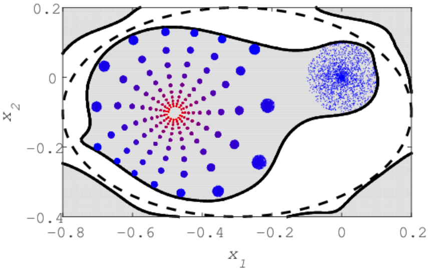

$\mathbf{X}_5^\infty$ with $c_1 = -0.7$ and $c_2 = -0.2$

Julia Map

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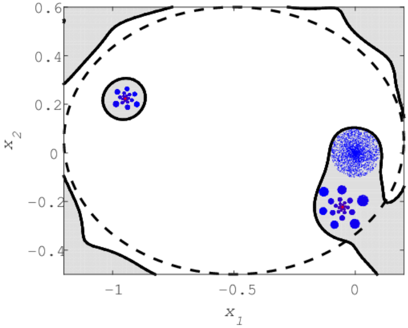

$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = 0.2$

Julia Map

Trajectories from $\mathbf{X}_0 := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 0.1^2\}$ under

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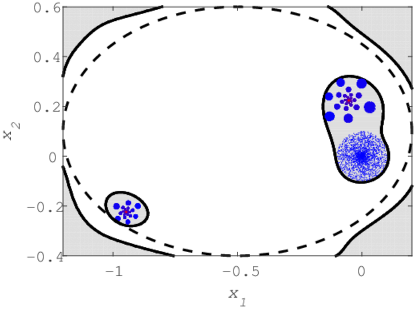


$\mathbf{X}_5^\infty$ with $c_1 = -0.9$ and $c_2 = -0.2$

Conclusion and Perspectives

$\oplus$ Certified Approximation of the **whole reachable set** $\mathbf{X}^\infty$

$\ominus$ Computational complexity: $\binom{n+2rd}{n}$ SDP variables

$\oplus$ **Structure sparsity** may be exploited

💡 Exploiting Sparsity for Volume Computation [Tacchi et al. 19]

Conclusion and Perspectives

Further research

- Volume Assumption: $\lim_{T \to \infty} \sum_{t=0}^{T} t \operatorname{vol} \mathbf{Y}^t \leqslant \infty$ always true?
- **Exact** certification: $\mathbf{X}_r^T = \{\mathbf{x} \in \mathbf{X} : v_r(\mathbf{x}) + u_r T \geqslant 0\} \supsetneq \mathbf{X}^T$

Perspectives: Exact Certificates

APPROXIMATE SOLUTIONS

sum of squares of $a^2 - 2ab + b^2$?

$(1.00001a - 0.99998b)^2$!

$$a^2 - 2ab + b^2 \simeq (1.00001a - 0.99998b)^2$$

$$a^2 - 2ab + b^2 \neq 1.0000200001a^2 - 1.9999799996ab + 0.9999600004b^2$$

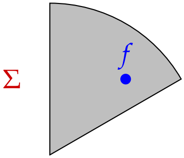
$\boxed{\simeq \quad \rightarrow \quad =}$?

Conclusion and Perspectives

Win TWO-PLAYER GAME

⇝ Univariate optimization [M.-Safey El Din-Schweighofer 18]

⇝ Multivariate optimization [M.-Safey El Din 18]



sum of squares of f?

≃ Output!

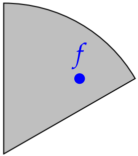
Conclusion and Perspectives

Win TWO-PLAYER GAME

⇝ Univariate optimization [M.-Safey El Din-Schweighofer 18]

⇝ Multivariate optimization [M.-Safey El Din 18]

Σ



Hybrid Symbolic/Numeric Algorithms

sum of squares of $f - \varepsilon$?

$\simeq$ Output!

Error Compensation

$\simeq \quad \rightarrow \quad =$

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Thank you for your attention!

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