Data-driven computation of the maximum positively invariant set for nonlinear dynamical systems

Milan Korda

(LAAS, CNRS)

Maximum positively invariant set

$$x^+ = f(x)\,, \quad x \in \mathbb{R}^n$$

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$f^{(k)}(x) = k^{\text{th}}$ iterate of the discrete recurrence starting from x

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MPI set

Set of all initial states that stay in the constraint set **X** forever

Maximum positively invariant set

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MPI set

$$\mathbf{X}^\star = \left\{ x \mid f^{(k)}(x) \in \mathbf{X} \quad \forall\, k \in \{0, 1, \ldots\} \right\}$$

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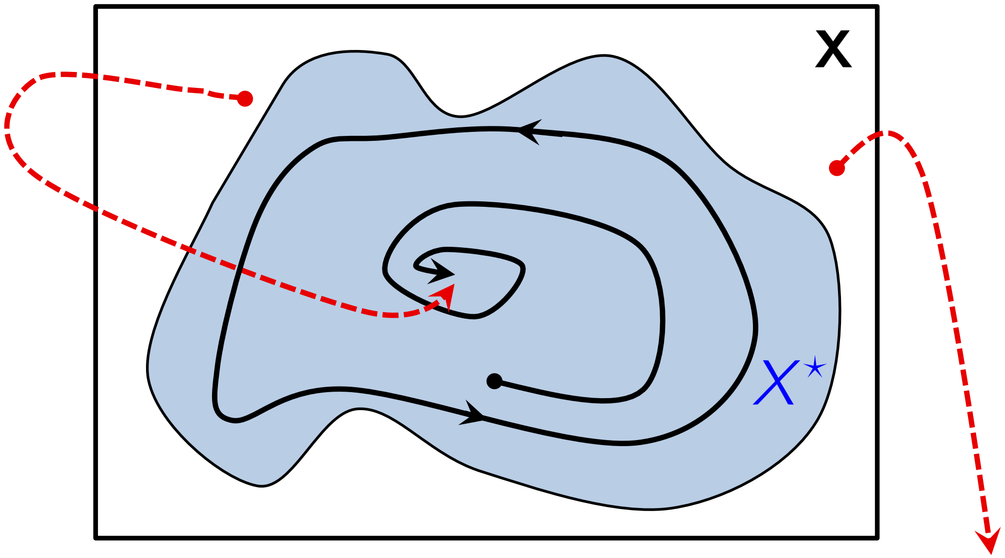


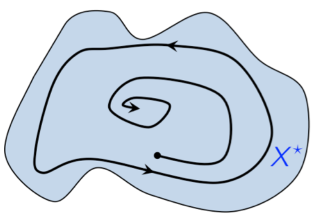
$$\min_{\mu} \langle g, \mu \rangle$$
$$\mathcal{A}\mu = b$$
$$\mu \in \mathcal{M}^+$$

$\Rightarrow$

$\Rightarrow$

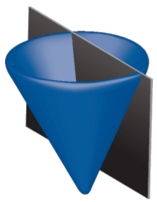
Linear program
(**Infinite**-dimensional)

$$x^+ = f(x)$$



Highly **nonconvex**

$$\min_{\boldsymbol{y}} \langle g_N, \boldsymbol{y} \rangle$$
$$\mathcal{A}_N \boldsymbol{y} = b_N$$
$$\boldsymbol{y} \in \mathcal{M}_N^+$$



Convex optimization problem

Primal LP

The MPI set is characterized by the optimization problem

Primal LP

$$\begin{aligned} \sup_{\mu,\mu_0} \quad & \int_X 1 \, d\mu_0 \\ \text{s.t.} \quad & \mu_0 + \alpha\, f_{\#}\mu - \mu = 0 \\ & \mu_0 \leq \lambda \\ & \mu \in \mathcal{M}(\mathbf{X})_+, \ \mu_0 \in \mathcal{M}(\mathbf{X})_+ \end{aligned}$$

Infinite dimensional **linear program** in the cone of nonnegative measures

[Korda, Henrion, Jones, 2014]

Dual LP on continuous functions

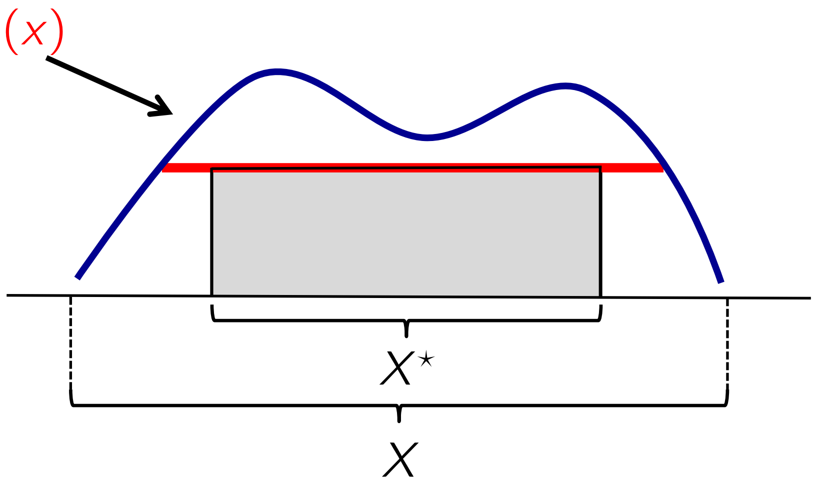
Dual LP

$$\begin{aligned}
\inf_{v,w} \quad & \int_{\mathbf{X}} w(x)\,dx \\
\text{s.t.} \quad & \alpha v(f(x)) \le v(x), && \forall x \in \mathbf{X} \\
& w(x) \ge v(x) + 1, && \forall x \in \mathbf{X} \\
& w(x) \ge 0 && \forall x \in \mathbf{X}
\end{aligned}$$

where the infimum is over $v \in C(\mathbf{X})$ and $w \in C(\mathbf{X})$

key obseravtion:

$$w \ge \mathbb{I}_{X^\star} \quad \Rightarrow \quad \{x \mid w(x) \ge 1\} \supset X^\star$$



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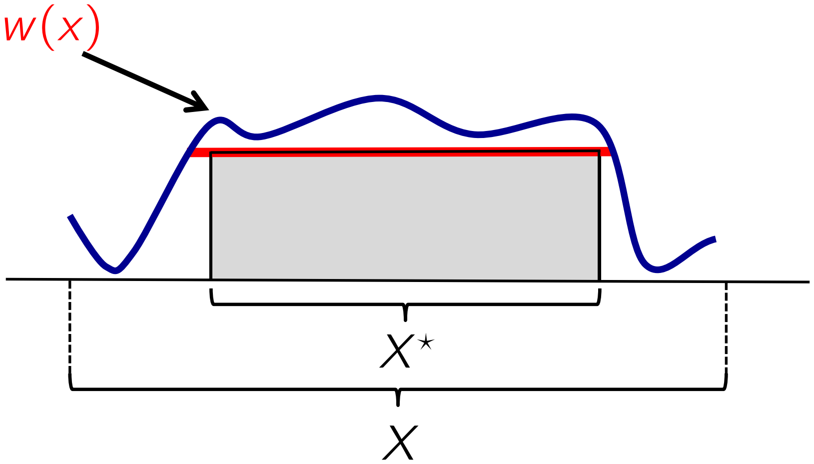

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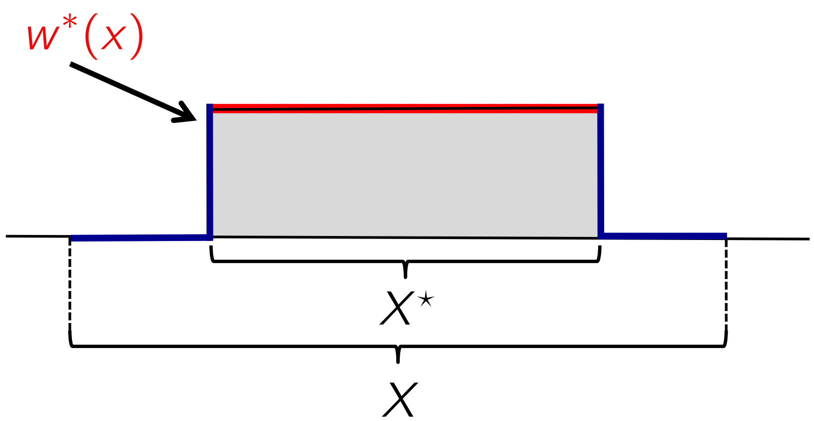


SDP hierarchy

Optimization over nonnegative continuous functions $\Leftrightarrow$ Optimization over nonnegative polynomials $\Rightarrow$ Optimization over SOS polynomials up to degree 2k

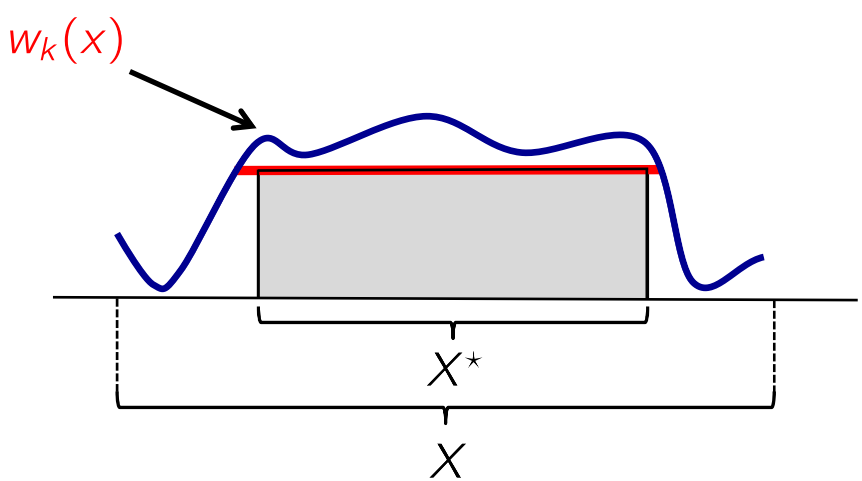
SDP

Convergence

Let $w_k(x)$ be the optimal solution to the dual SDP relaxation of order k

$$X_k^\star := \{x \mid w_k(x) \geq 1\}$$

$$\mathrm{vol}(X_k^\star \setminus X^\star) \to 0$$



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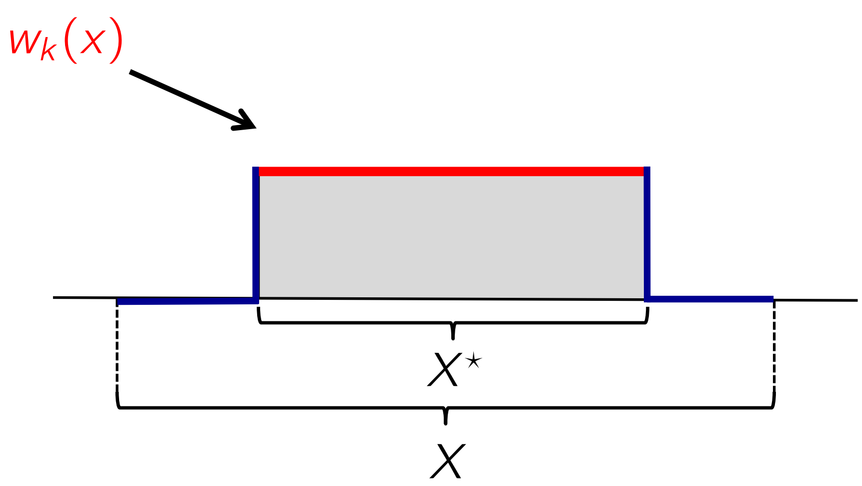
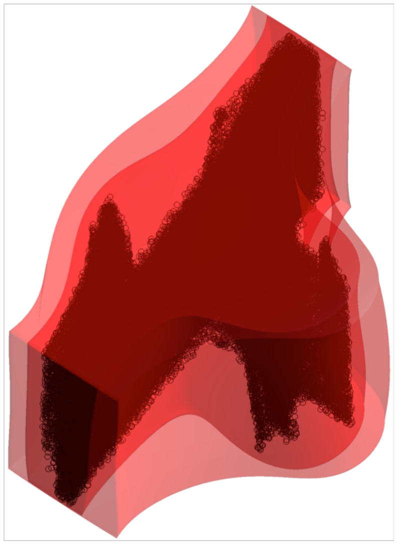
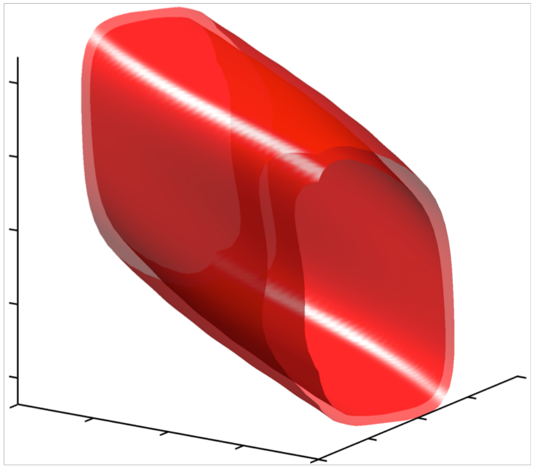


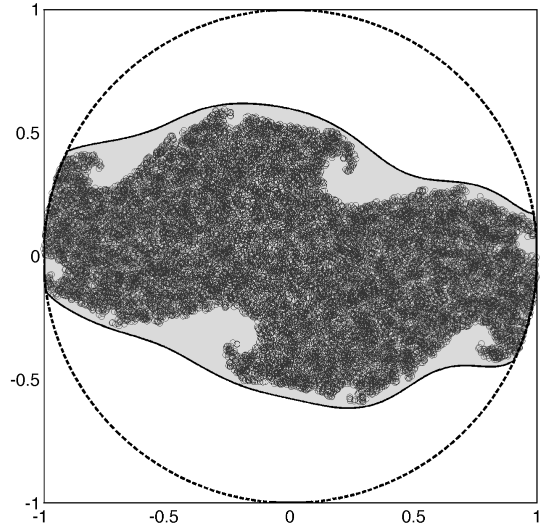
3D Hénon



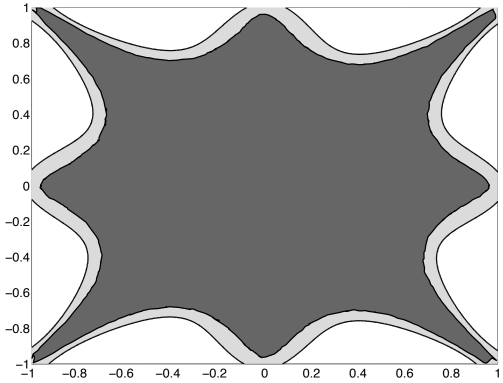
Double pendulum on cart



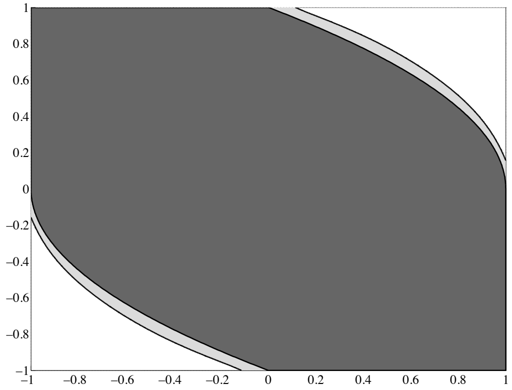
Julia



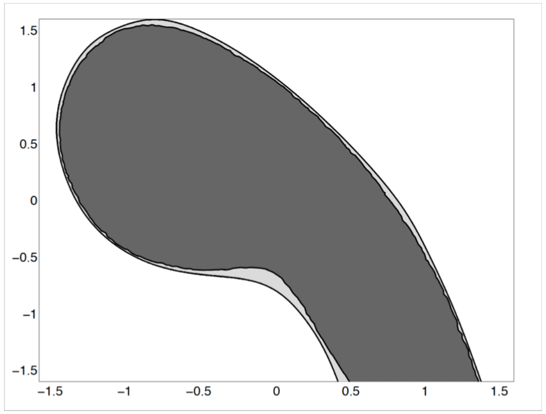
Spider web



Double integrator



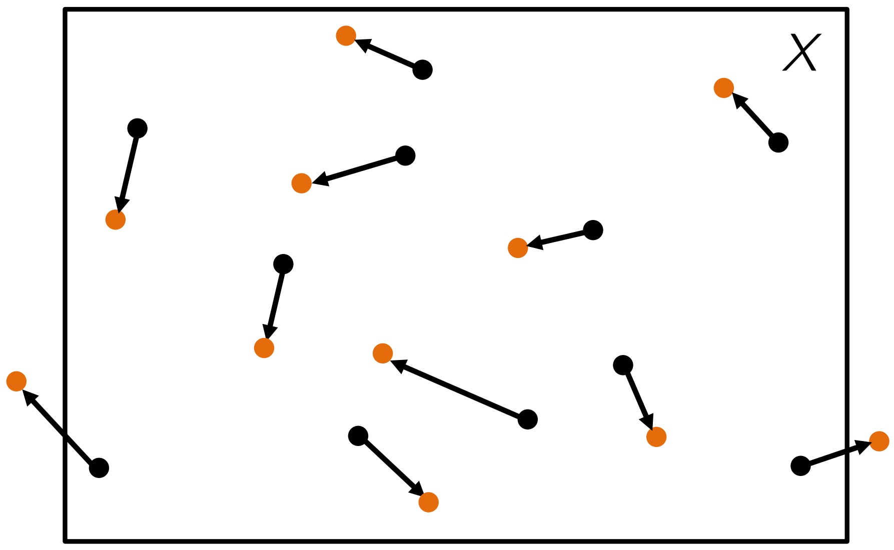
Cathala



Data-driven invariant set estimation

Maximum positively invariant set from data

f **not given**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



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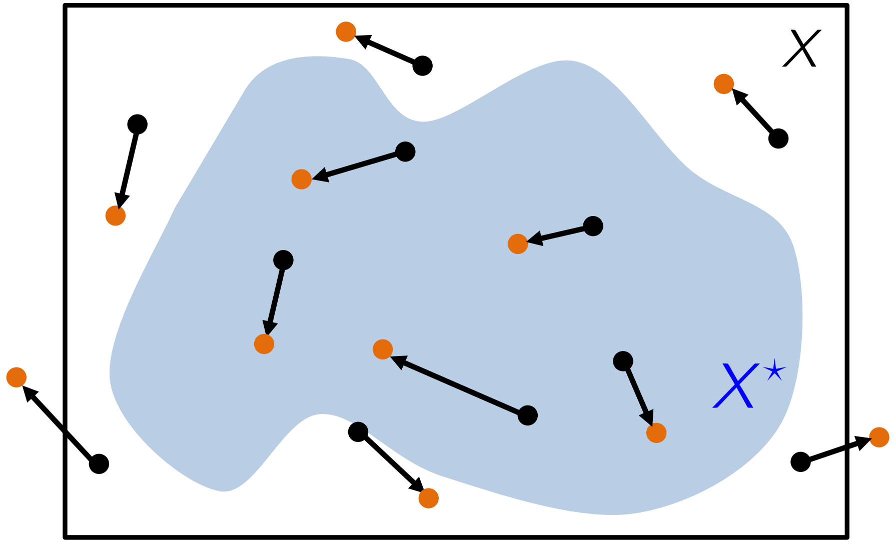

First idea

Sampled dual LP

$$
\begin{aligned}
\inf \quad & \frac{1}{M}\sum_{i=1}^{M} w(x_i) \\
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\end{aligned}
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with variables $v, w \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

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Properties

- **+** **No assumptions** on f (can be non-polynomial, discontinuous etc.)
- **+** **No assumptions** on the subspace $\mathcal{F}$ (can be radial basis functions, wavelets etc.)
- **+** Boils down to **finite-dimensional LP**

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- – No longer guaranteed outer approximation

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Properties

- \+ **No assumptions** on f (can be non-polynomial, discontinuous etc.)
- \+ **No assumptions** on the subspace $\mathcal{F}$ (can be radial basis functions, wavelets etc.)
- \+ Boils down to **finite-dimensional LP**
- − No longer guaranteed outer approximation
- − Convergence rate and sample complexity hard to analyze

Where's the problem?

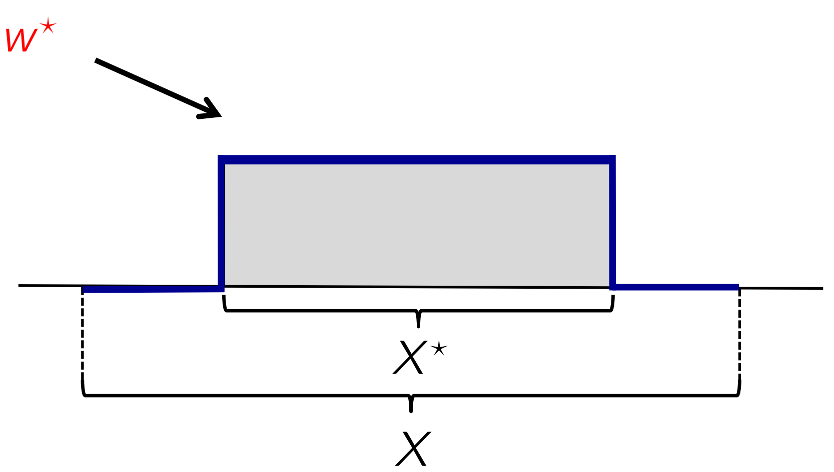
Dual LP

$$\begin{array}{lll} \inf\limits_{v,w} & \int_{\mathbf{X}} w(x)\,dx & \\ \text{s.t.} & \alpha v(f(x)) \le v(x), & \forall x \in \mathbf{X} \\ & w(x) \ge v(x) + 1, & \forall x \in \mathbf{X} \\ & w(x) \ge 0 & \forall x \in \mathbf{X} \end{array}$$

Infimum **not attained** in $C(\mathbf{X})$

$w^\star$ discontinuous

v growing unbounded



Solution to the problem: new LP formulation

$$\sup_{v} \quad \int_{X} v(x)\, dx$$

$$\text{s.t.} \quad v \le \ell + \alpha v \circ \bar{f} \quad \text{on} \quad \mathbf{X}$$

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Supremum attained by a $v^\star \in C(\mathbf{X})$ given by

$$v^\star(x) = \min \sum_{k=0}^{\infty} \alpha^k \ell(x_k)$$
$$\text{s.t.} \quad x_{k+1} = \bar{f}(x_k)$$
$$x_k \in \mathbf{X}$$
$$x_0 = x$$

if
- ℓ and $\bar{f}$ are continuous
- $\bar{f}(\mathbf{X}) \subset \mathbf{X}$ (= $\mathbf{X}$ positively invariant under $\bar{f}$)

Solution to the problem: new LP formulation

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Supremum attained by a $v^\star \in C(\mathbf{X})$ given by

$$\begin{aligned} v^\star(x) \quad = \quad \min \quad & \textstyle\sum_{k=0}^{\infty} \alpha^k \ell(x_k) \\ \text{s.t.} \quad & x_{k+1} = \bar{f}(x_k) \\ & x_k \in \mathbf{X} \\ & x_0 = x \end{aligned}$$

Strategy

Find **continuous** ℓ and $\bar{f}$ satisfying $\bar{f}(\mathbf{X}) \subset \mathbf{X}$ such that

- $\{x \mid v^\star(x) \le 0\} = \mathbf{X}^\star$
- $v \le \ell + \alpha v \circ \bar{f}$ can be evaluated on samples $(x, f(x))$

New LP

$$\sup_{v} \int_X v(x)\, dx$$

$$\text{s.t.} \quad v \le \text{dist}_X \circ f + \alpha v \circ \text{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

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Crucial fact

$$v^\star(x) \;=\; \min \;\; \sum_{k=0}^{\infty} \alpha^k \ell(x_k)$$

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Remark: $\mathbf{X}$ invariant under $\bar{f}$

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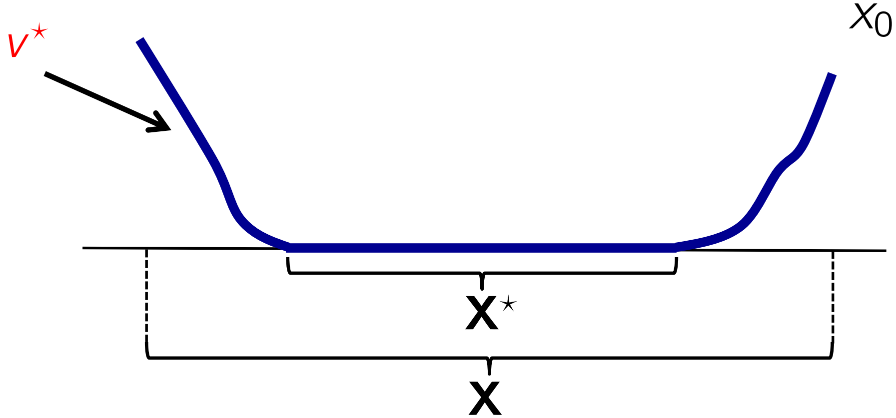

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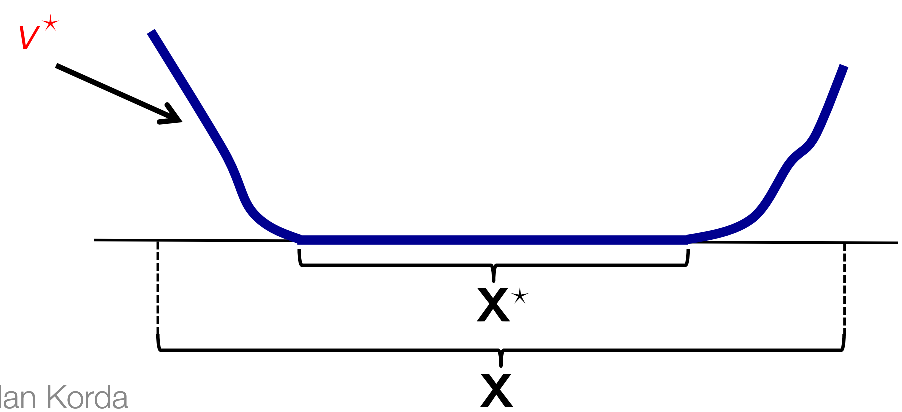

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Lemma

$$\alpha < \frac{1}{\mathrm{Lip}\, f} \quad \Rightarrow \quad v^\star \text{ Lipschitz} \quad \text{with} \quad \mathrm{Lip}\, v^\star \le \frac{1}{1 - \alpha \cdot \mathrm{Lip}\, f}$$

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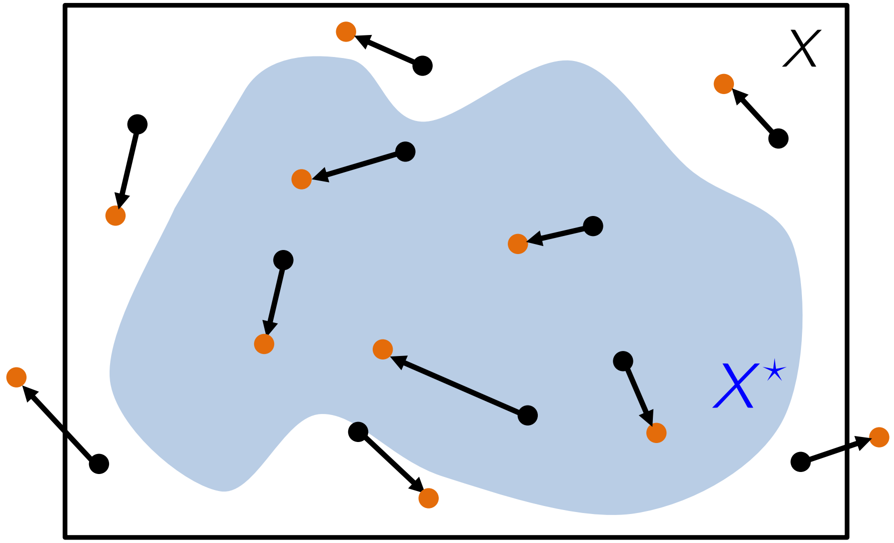
$$\begin{cases} = 0 & \text{if} \quad x \in \mathbf{X}^\star \\ > 0 & \text{if} \quad x \notin \mathbf{X}^\star \end{cases}$$

Fact

v feasible $\Rightarrow$ $\{x \mid v(x) \leq 0\} \supset \mathbf{X}^\star$

Data-driven approximation

f **not given**, only **data** $\{x_i, x_i^+\}_{i=1}^K$ available



Sampled LP

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with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

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with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

Properties

+ **No assumptions** on f (can be non-polynomial, discontinuous* etc.)
+ **No assumptions** on the subspace $\mathcal{F}$ (can be radial basis functions, wavelets etc.)
+ Boils down to **finite-dimensional LP**
− No longer guaranteed outer approximation
+ Can analyze **convergence rate** and sample **complexity**

Convergence rate

$$\sup_{v} \int_{X} v(x)\, dx$$

$$\text{s.t.} \quad v \le \mathrm{dist}_X \circ f + \alpha v \circ \mathrm{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

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Key dynamic programming estimate (*Farias, van Roy, 2003*)

$$\int_{\mathbf{X}} |v^\star - v_{\mathcal{F}}|\,dx \le \frac{1}{1-\alpha} \min_{v \in \mathcal{F}} \|v^\star - v\|_\infty$$

Convergence rate

$$\sup_{v} \int_X v(x)\, dx$$

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Example: $\mathcal{F}$ = multivariate polynomials up to degree d

$$\int_X |v^\star - v_{\mathcal{F}}| \le \frac{c}{(1-\alpha)(1-\alpha \mathrm{Lip}(f))} \frac{1}{d}$$

Convergence rate

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with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathrm{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}_{\infty}) \leq ??$

Convergence rate

$$\sup_{v} \int_{X} v(x)\, dx$$

$$\text{s.t.} \quad v \leq \mathrm{dist}_X \circ f + \alpha v \circ \mathrm{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}^{\star}) \leq \frac{c}{(1-\alpha)(1-\alpha \mathrm{Lip}(f))} \frac{1}{\sqrt{d}} + g_{v^{\star}}\Big(\frac{1}{\sqrt{d}}\Big)$$

$$g_{v^{\star}}(\gamma) = \mathrm{vol}(\{x \mid 0 < v^{\star}(x) \leq \gamma\})$$

Convergence rate

$$\sup_{v} \int_X v(x)\,dx$$

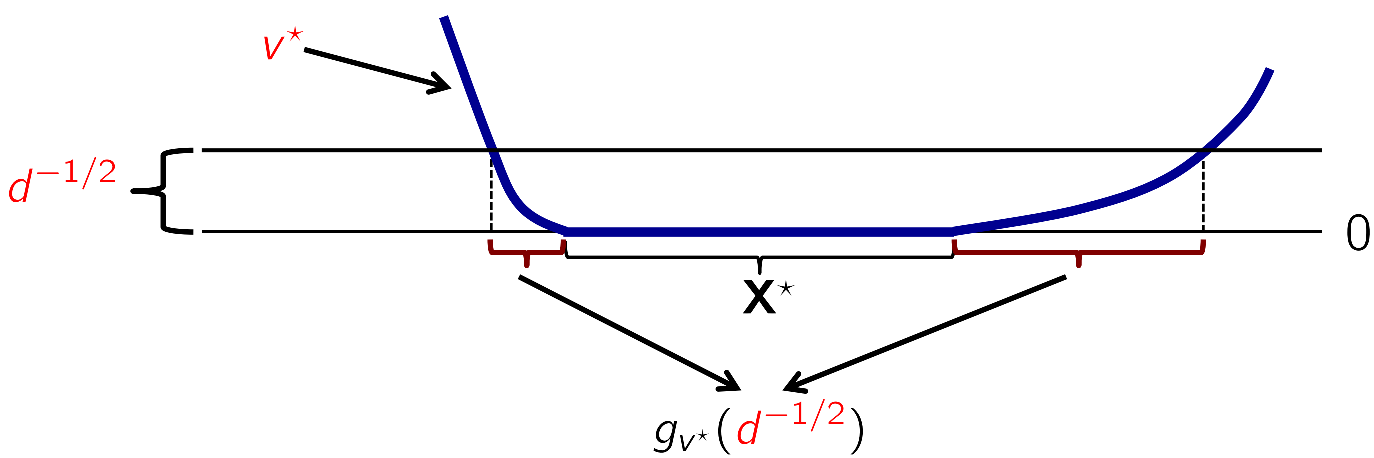
$$\text{s.t.} \quad v \le \mathrm{dist}_X \circ f + \alpha v \circ \mathrm{proj}_X \circ f \quad \text{on} \quad \mathbf{X}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F}} \setminus \mathbf{X}^\star) \le \frac{c}{(1-\alpha)(1-\alpha\mathrm{Lip}(f))} \frac{1}{\sqrt{d}} + g_{v^\star}\Big(\frac{1}{\sqrt{d}}\Big)$$

$$g_{v^\star}(d^{-1/2}) = \mathrm{vol}\big(\{x \mid 0 < v^\star(x) \le d^{-1/2}\}\big)$$



Sample complexity

$$\begin{aligned} \inf_{v} \quad & \int v(x)\,dx \\ \text{s.t.} \quad & \alpha v(x_i) \le \text{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\text{proj}_{\mathbf{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned} \Bigg\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \text{Data} \\ & |\text{Data}| = K \end{aligned}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\left| \int_X v_{\mathcal{F},K} - \int_X v_{\mathcal{F}} \right| < \epsilon$$

with probability at least $1 - \delta$ if

$$K \ge \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{\mathbf{X},\mathcal{F}}}{\epsilon(1-\alpha)})}{\log\left(\frac{1}{1-\left[\frac{\epsilon(1-\alpha)}{L_{\mathbf{X},\mathcal{F}}}\right]^n}\right)}$$

Sample complexity

$$\begin{aligned} \inf_{v} \quad & \int v(x)\, dx \\ \text{s.t.} \quad & \left. \begin{aligned} & \alpha v(x_i) \leq \mathrm{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathbf{X}}(x_i^+)) \\ & -1 \leq v(x_i) \leq (1-\alpha)^{-1} \end{aligned} \right\} \quad \begin{aligned} & \forall (x_i, x_i^+) \in \mathrm{Data} \\ & |\mathrm{Data}| = K \end{aligned} \end{aligned}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\left| \int_X v_{\mathcal{F},K} - \int_X v_{\mathcal{F}} \right| < \epsilon$$

with probability at least $1 - \delta$ if

$$K \geq \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{\mathbf{X},\mathcal{F}}}{\epsilon(1-\alpha)})}{\log\left(\frac{1}{1-\left[\frac{\epsilon(1-\alpha)}{L_{\mathbf{X},\mathcal{F}}}\right]^n}\right)} \approx \frac{\log(\frac{1}{\delta}) + n \log(\frac{L_{\mathbf{X},\mathcal{F}}}{\epsilon(1-\alpha)})}{\left[\frac{\epsilon(1-\alpha)}{L_{\mathbf{X},\mathcal{F}}}\right]^n}$$

Sample complexity

$$\begin{aligned} \inf_{v} \quad & \int v(x)\,dx \\ \text{s.t.} \quad & \left.\begin{aligned} &\alpha v(x_i) \le \mathrm{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathbf{X}}(x_i^+)) \\ &-1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned}\right\} \quad \begin{aligned} &\forall (x_i, x_i^+) \in \mathrm{Data} \\ &|\mathrm{Data}| = K \end{aligned} \end{aligned}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F},K} \setminus \mathbf{X}^{\star}) \le ??$$

Sample complexity

$$\begin{aligned} \inf_{v} \quad & \int v(x)\, dx \\ \text{s.t.} \quad & \alpha v(x_i) \le \mathrm{dist}_{\mathbf{X}}(x_i^+) + \alpha v(\mathrm{proj}_{\mathbf{X}}(x_i^+)) \\ & -1 \le v(x_i) \le (1-\alpha)^{-1} \end{aligned} \Bigg\} \quad \begin{aligned} &\forall (x_i, x_i^+) \in \mathrm{Data} \\ &|\mathrm{Data}| = K \end{aligned}$$

with the variable $v \in \mathcal{F} \subset \mathcal{C}(\mathbf{X})$, $\dim(\mathcal{F}) < \infty$

$\mathcal{F}$ = multivariate polynomials up to degree d

$$\mathrm{vol}(\mathbf{X}_{\mathcal{F},K} \setminus \mathbf{X}^\star) \le \frac{C}{K^{1/2n}} + g_{v^\star}\left(\frac{1}{K^{1/2n}}\right)$$

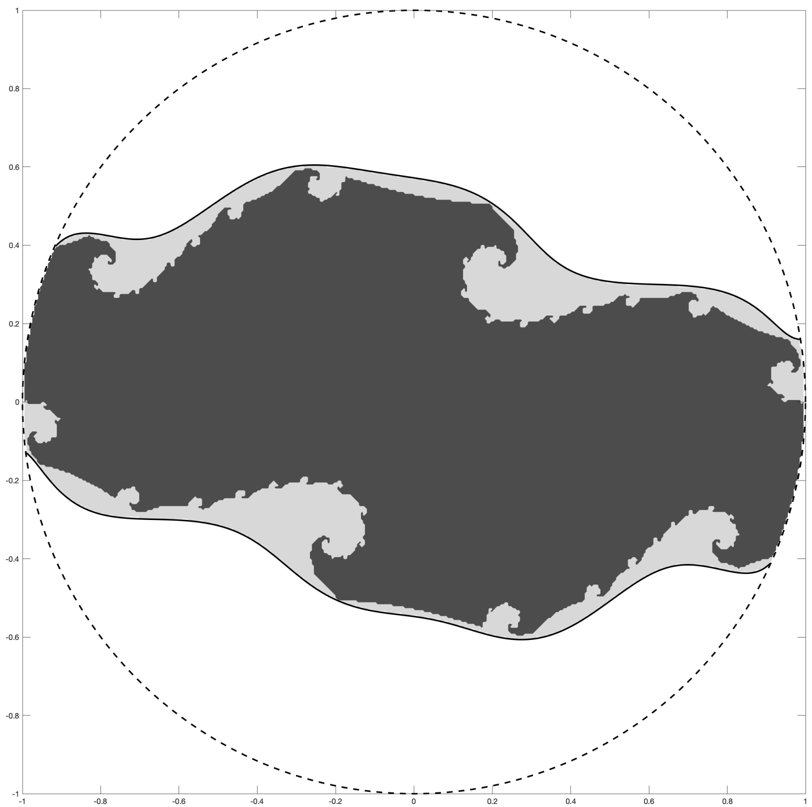
with probability at least $1-\delta$ if

Numerical examples

Julia set – sampling vs SDP

Basis: polynomials up to degree 10

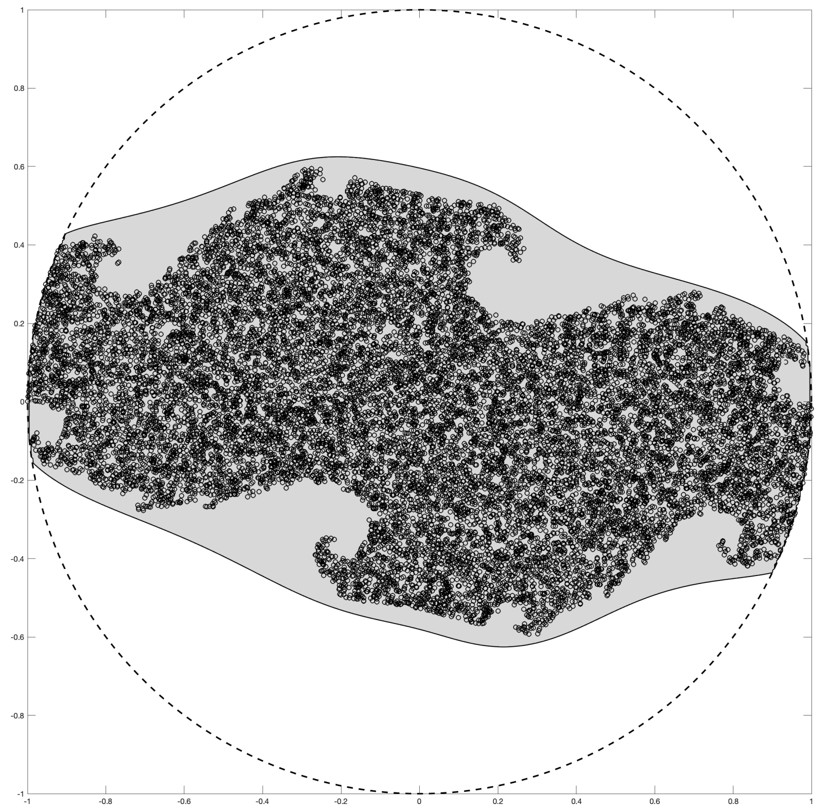
Sampling



Volume error 20.31 %

Misclassification 0 %

SDP



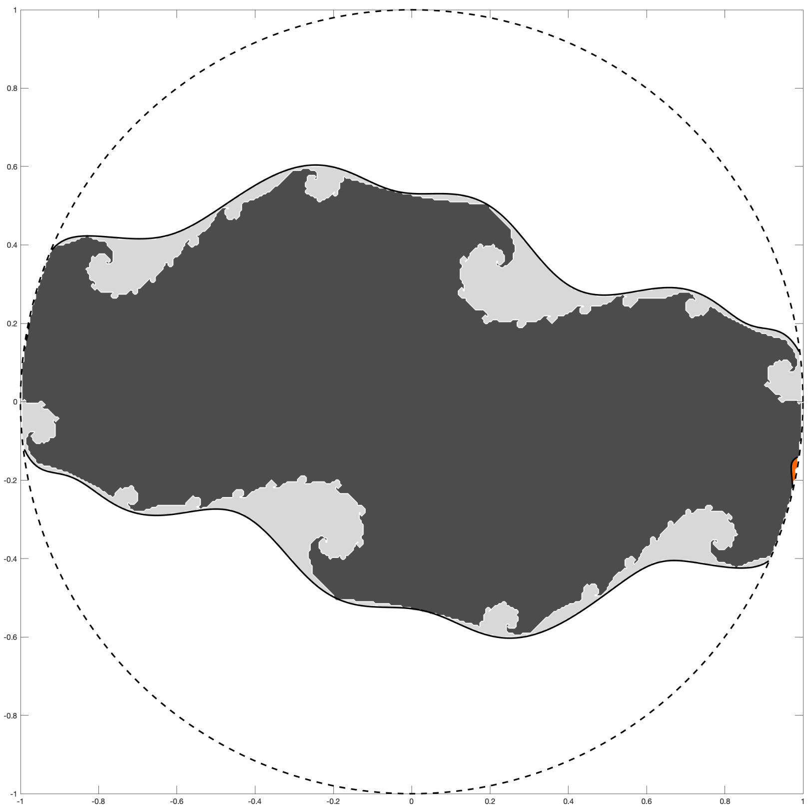
Volume error 28.7 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 14

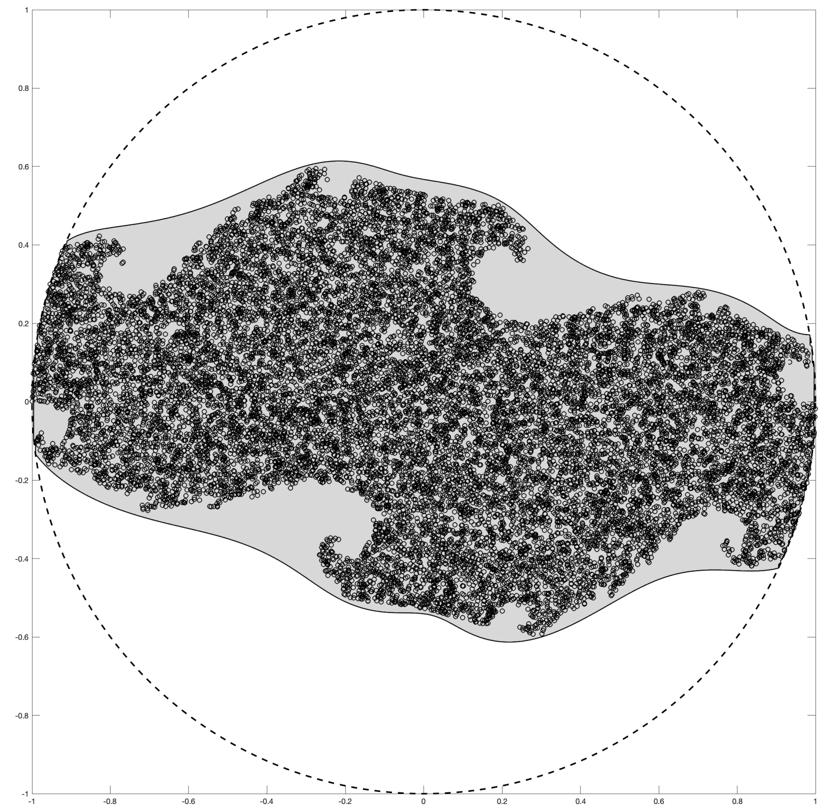
Sampling



Volume error 14.98 %

Misclassification 0.086 %

SDP



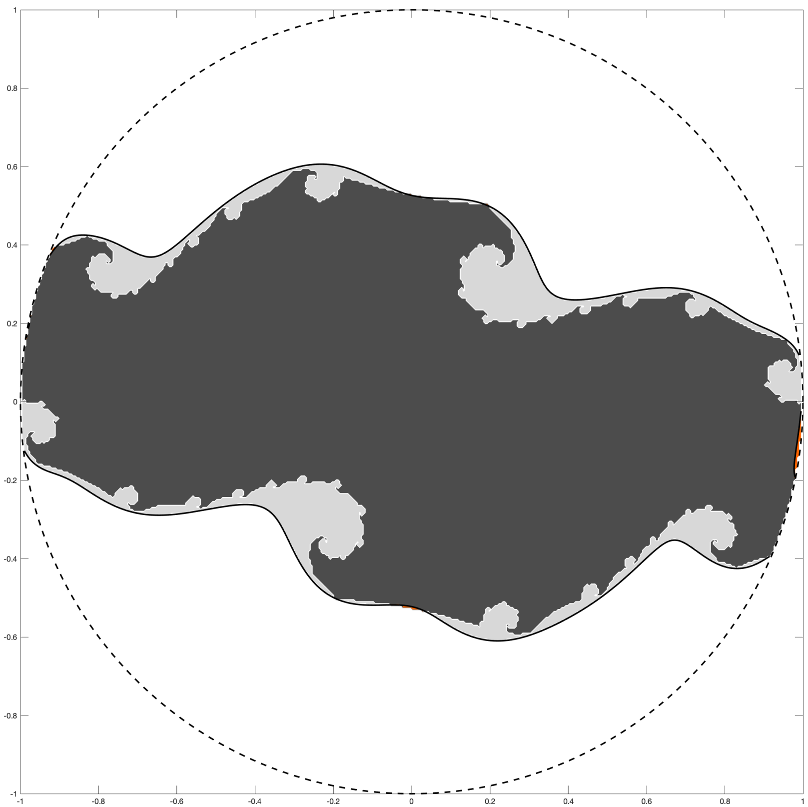
Volume error 21.9 %

Misclassification 0 %

Julia set – sampling vs SDP

Basis: polynomials up to degree 18

Sampling



Volume error 13.24 %

Misclassification 0.157 %

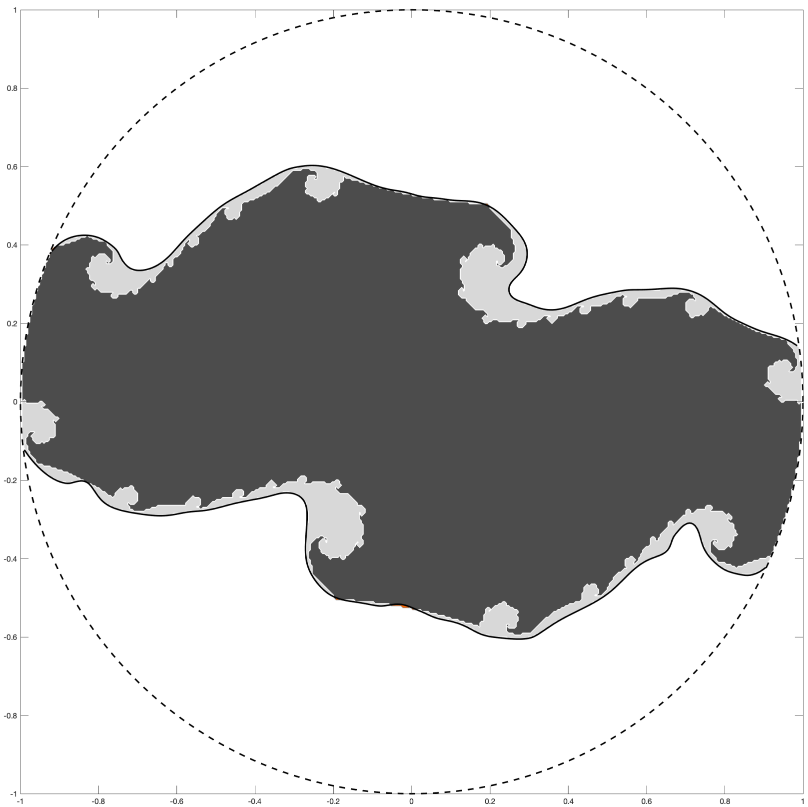
SDP

Numerical problems

Julia set – different bases

Basis: 400 thin-plate spline RBFs

Sampling



Volume error 10.78 %

Misclassification 0.041 %

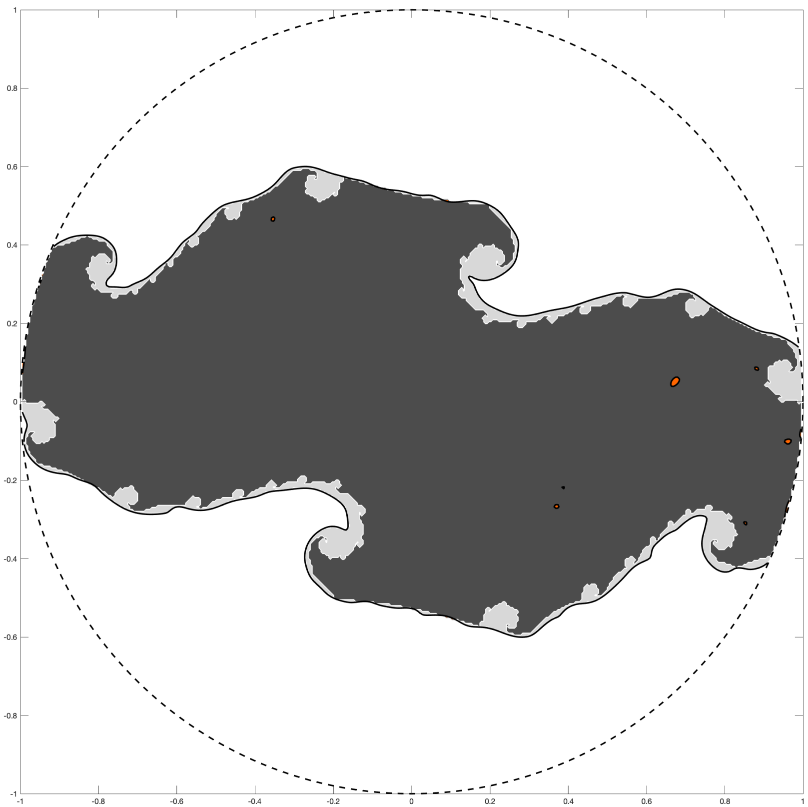
SDP

NA

Julia set – different bases

Basis: 1000 thin-plate spline RBFs

Sampling



Volume error 7.35 %

Misclassification 0.014 %

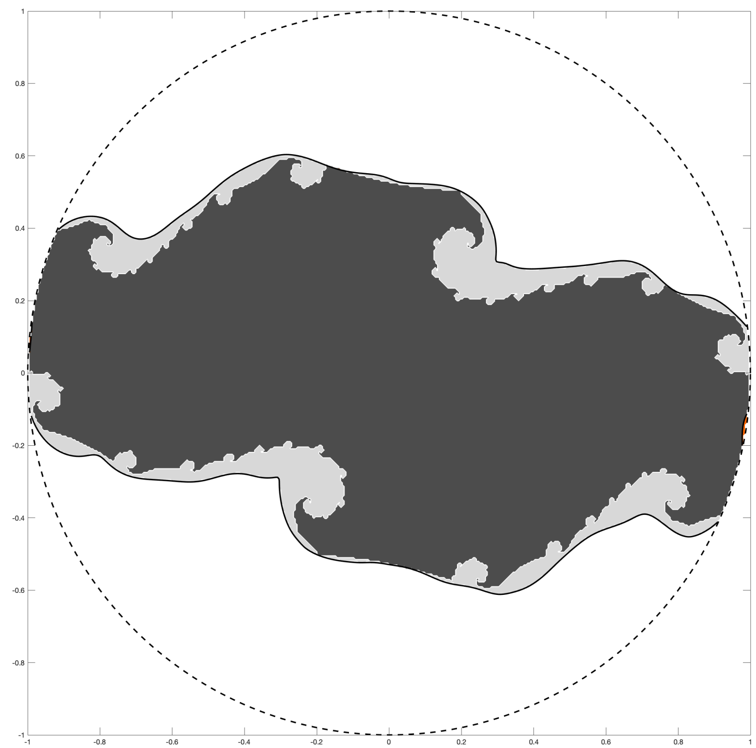
SDP

NA

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 15000



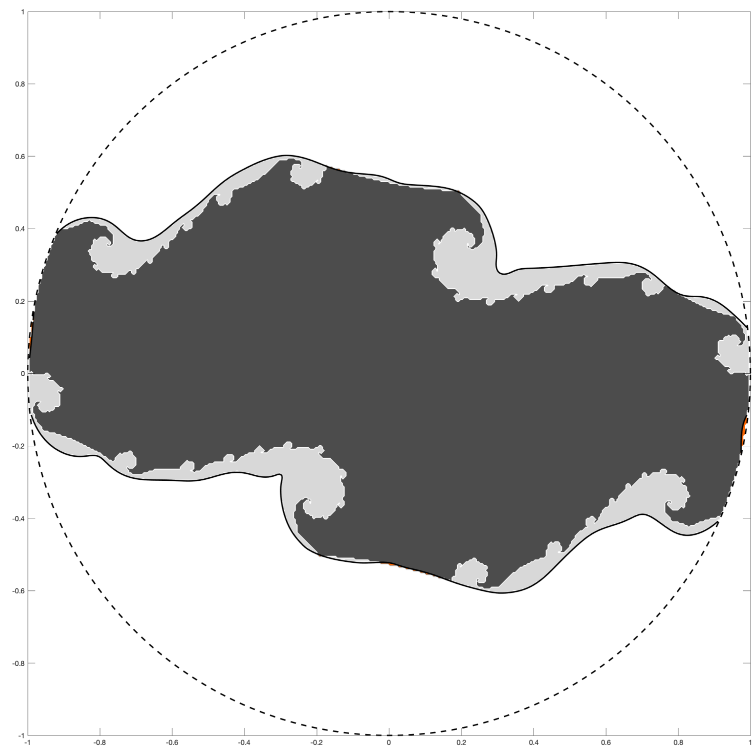
Volume error 15.1 %

Misclassification 0.09 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 10000



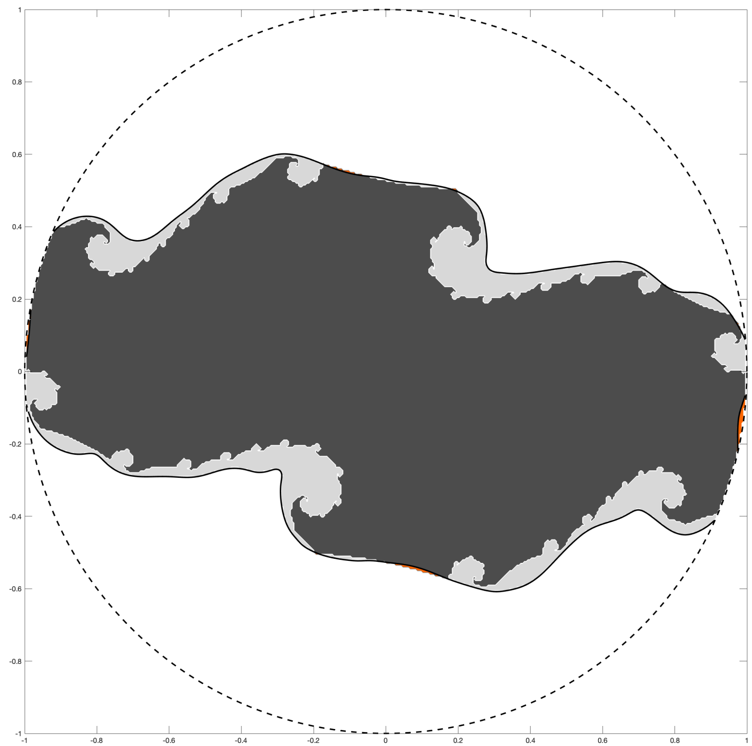
Volume error 14.5 %

Misclassification 0.19 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 5000



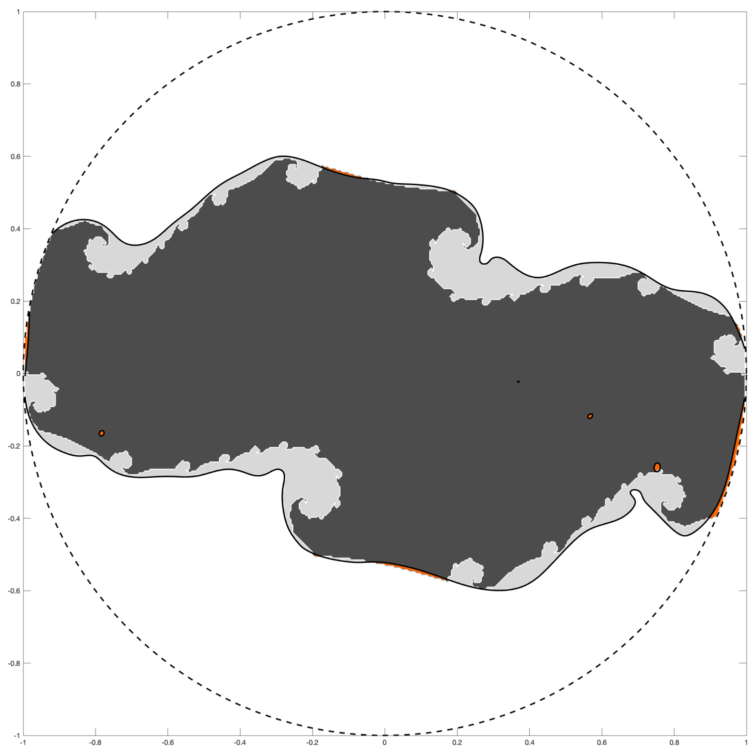
Volume error 13.5 %

Misclassification 0.296 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 3000



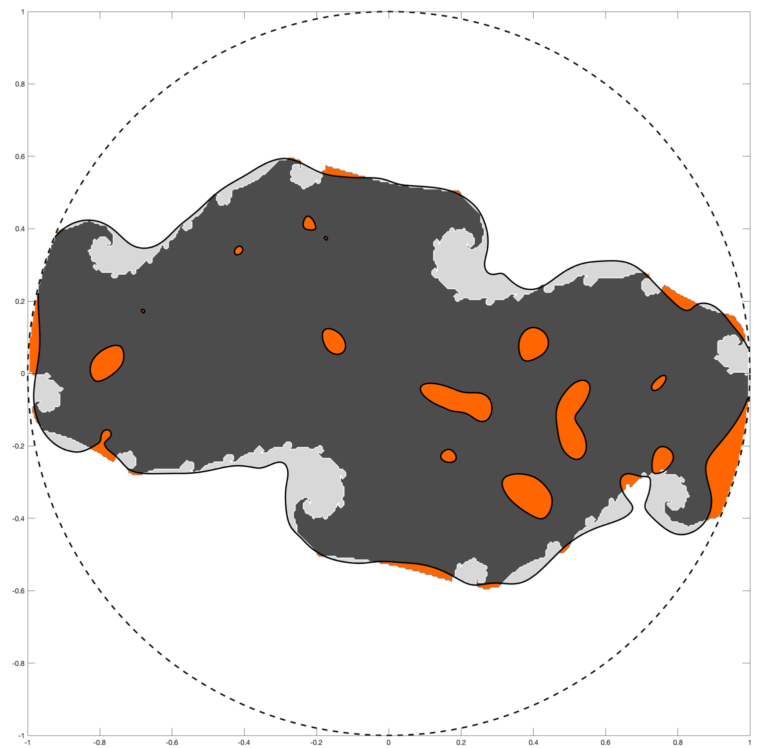
Volume error 12.65 %

Misclassification 0.59 %

Julia set – # samples

Basis: 200 thin-plate spline RBFs

 # Samples: 1000



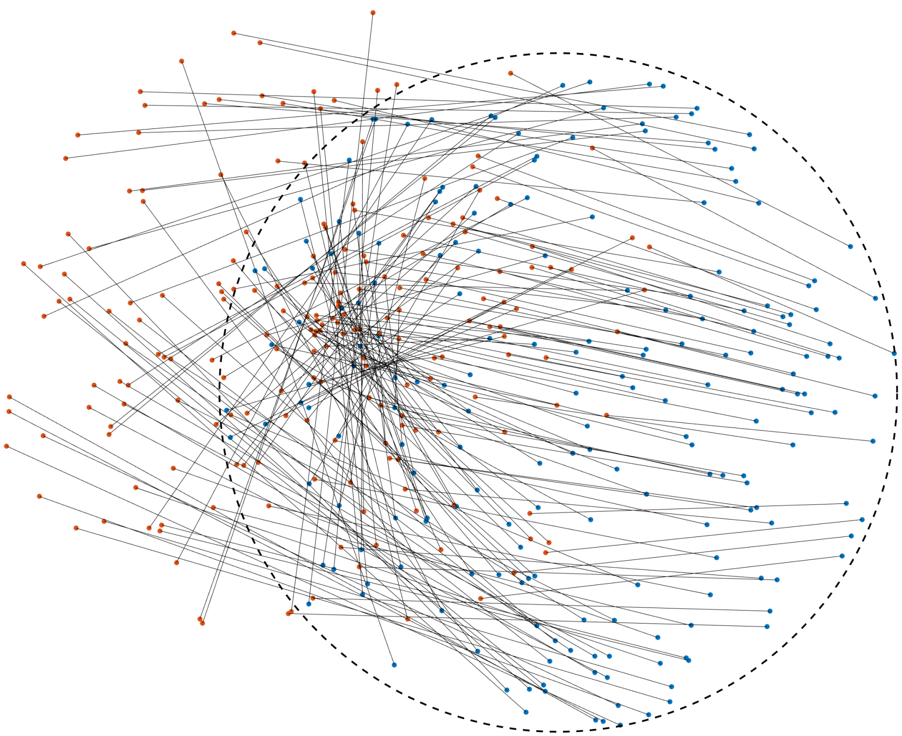
Volume error 10.41 %

Misclassification 6.84 %

Julia set – low data limit

 # Samples: 200

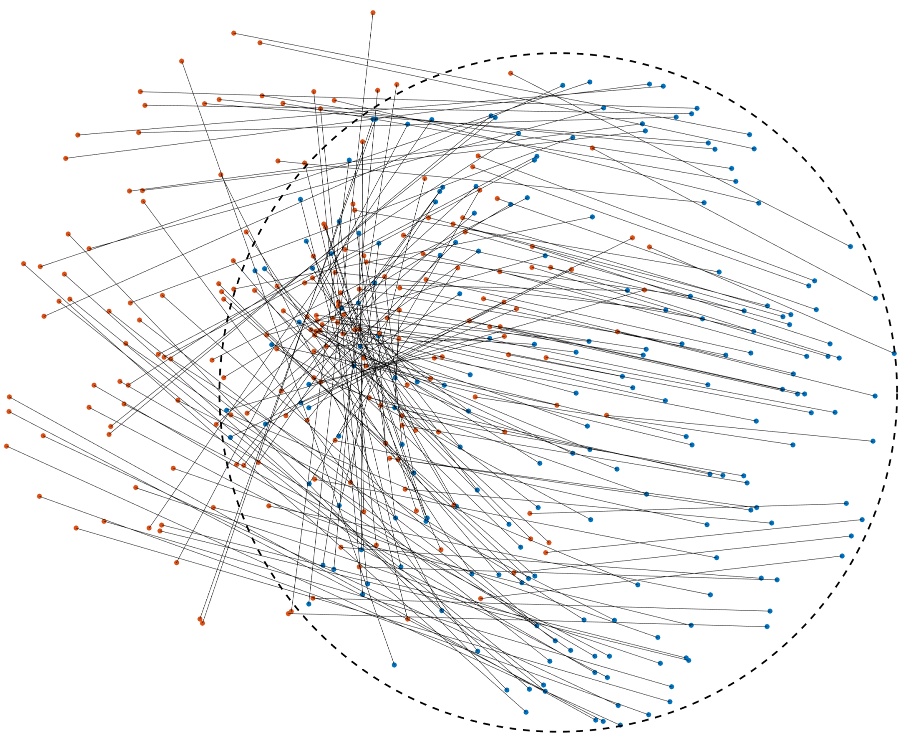
Data



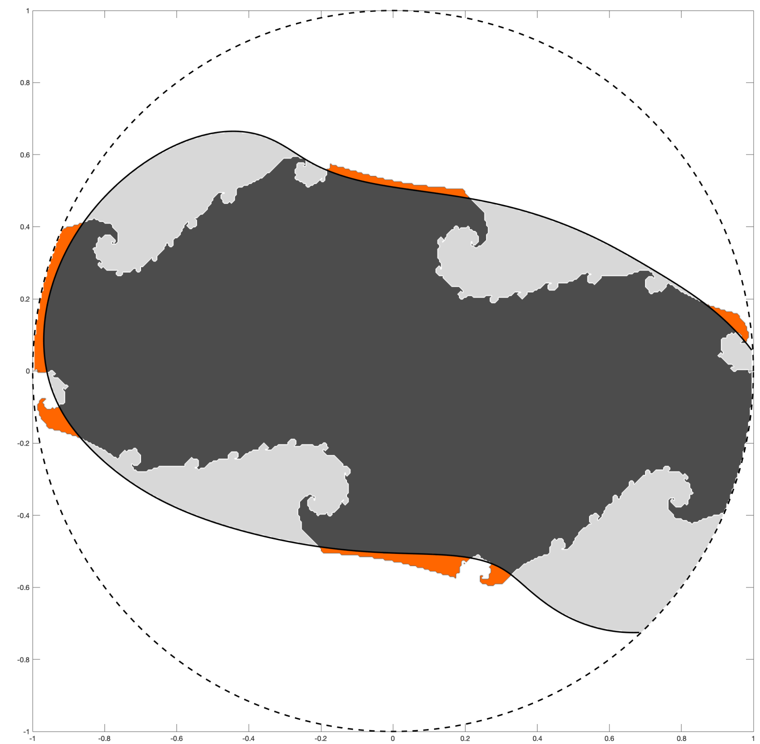
Julia set – low data limit

 # Samples: 200

Data



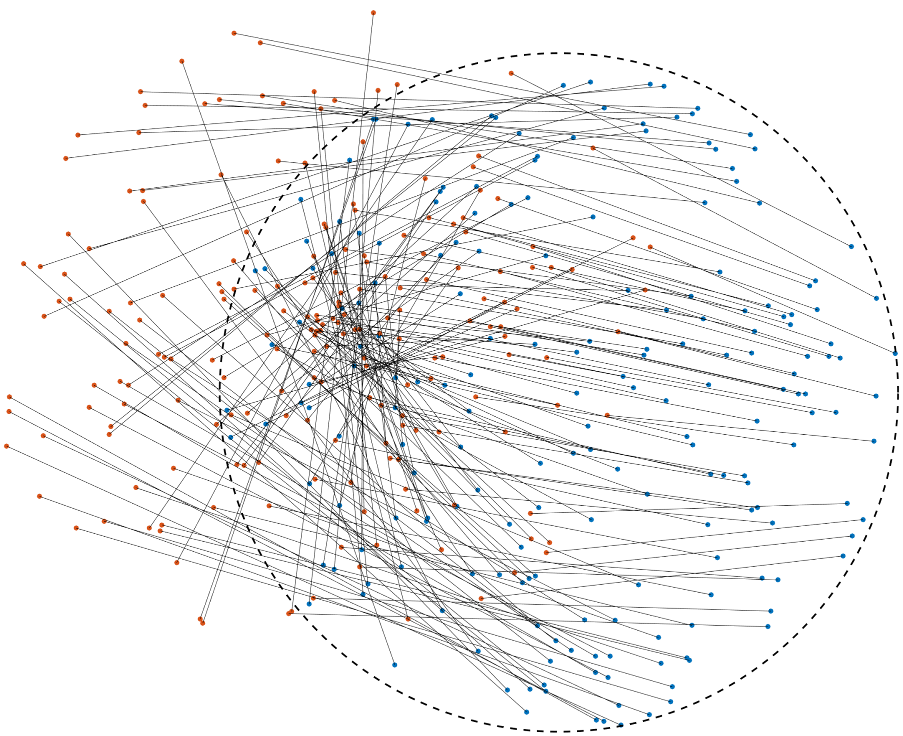
Approximation using 15 RBFs



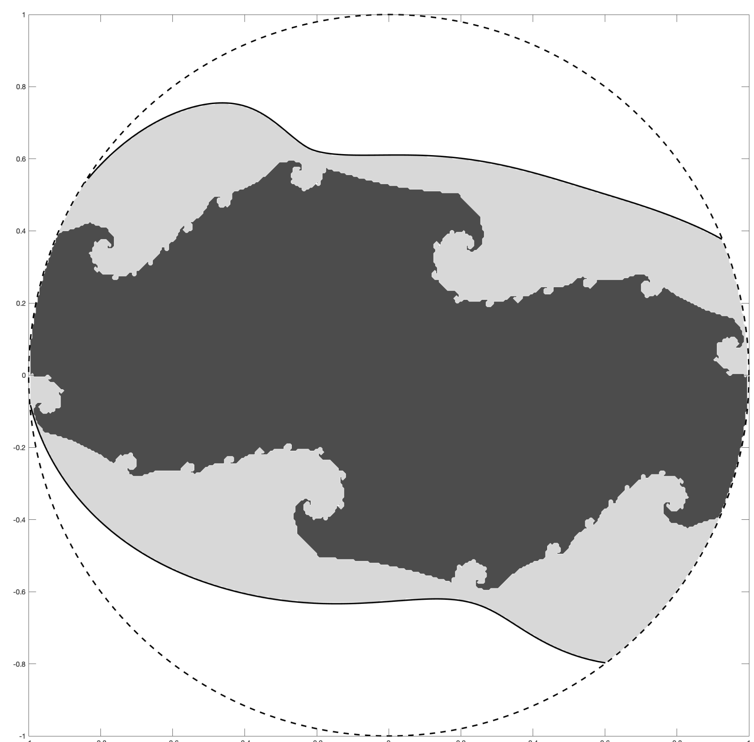
Numerics – Julia set – low data limit

 # Samples: 200

Data



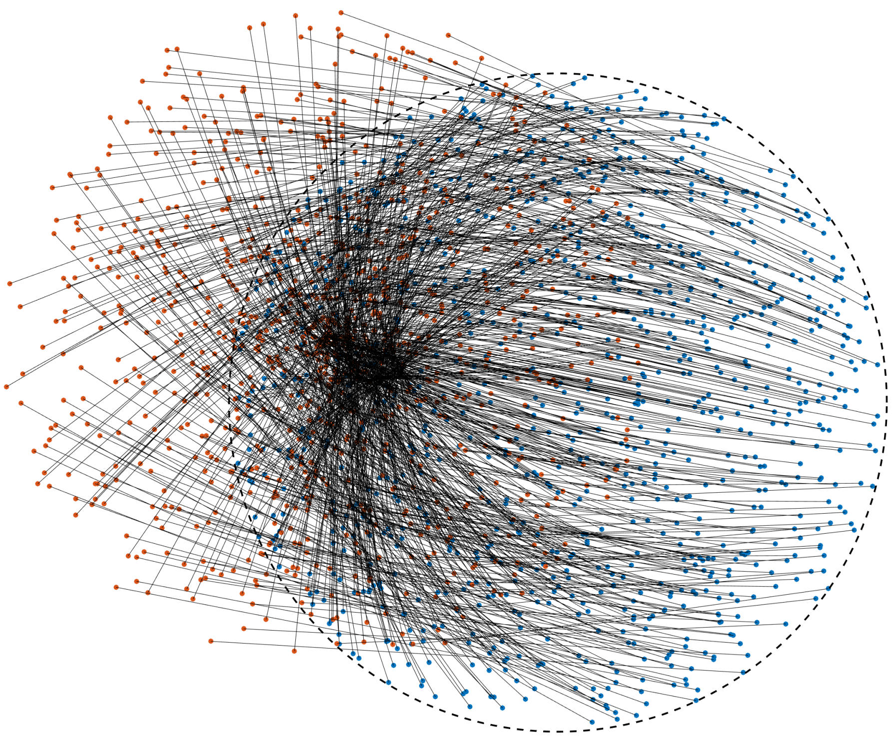
Approximation using 10 RBFs



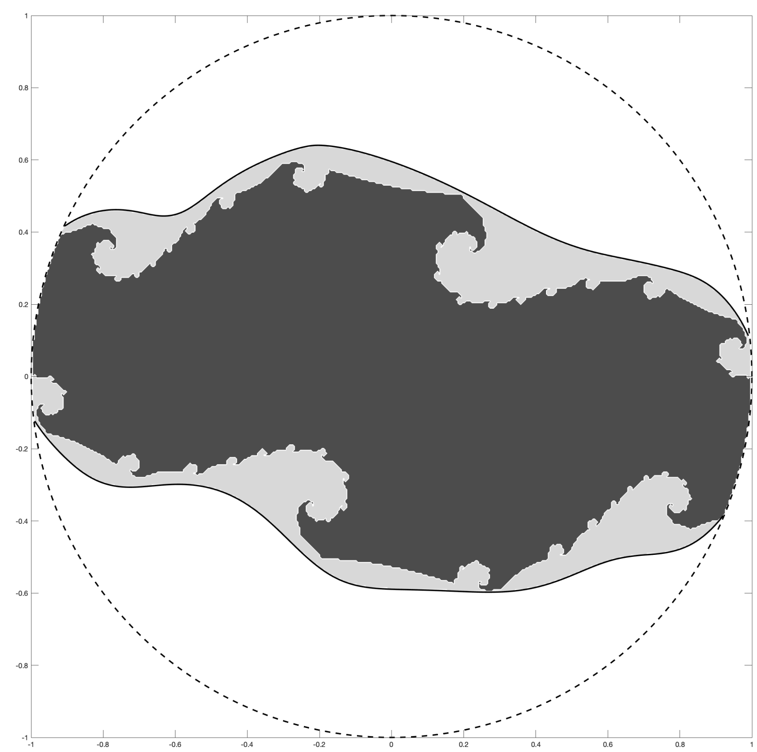
Numerics – Julia set – low data limit

 # Samples: 1000

Data

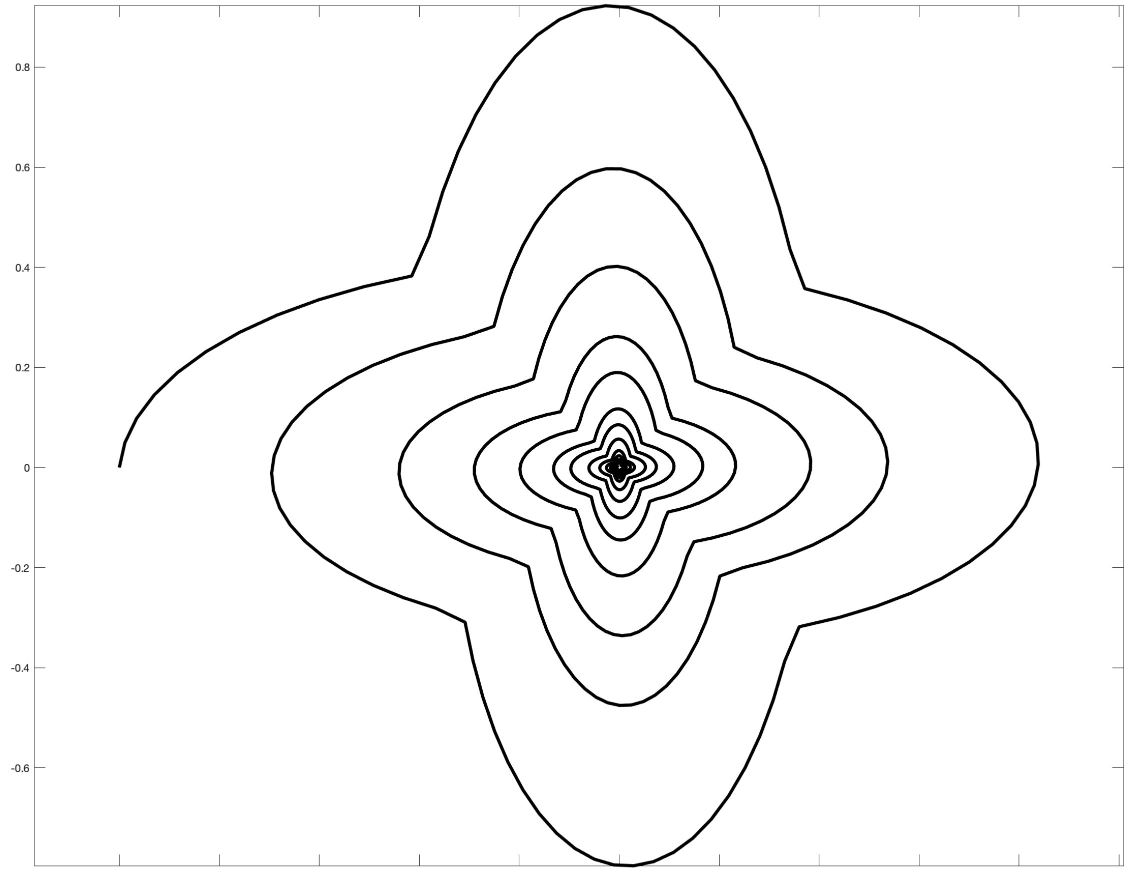


Approximation using 30 RBFs



Switched system

Flower system $$\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$$

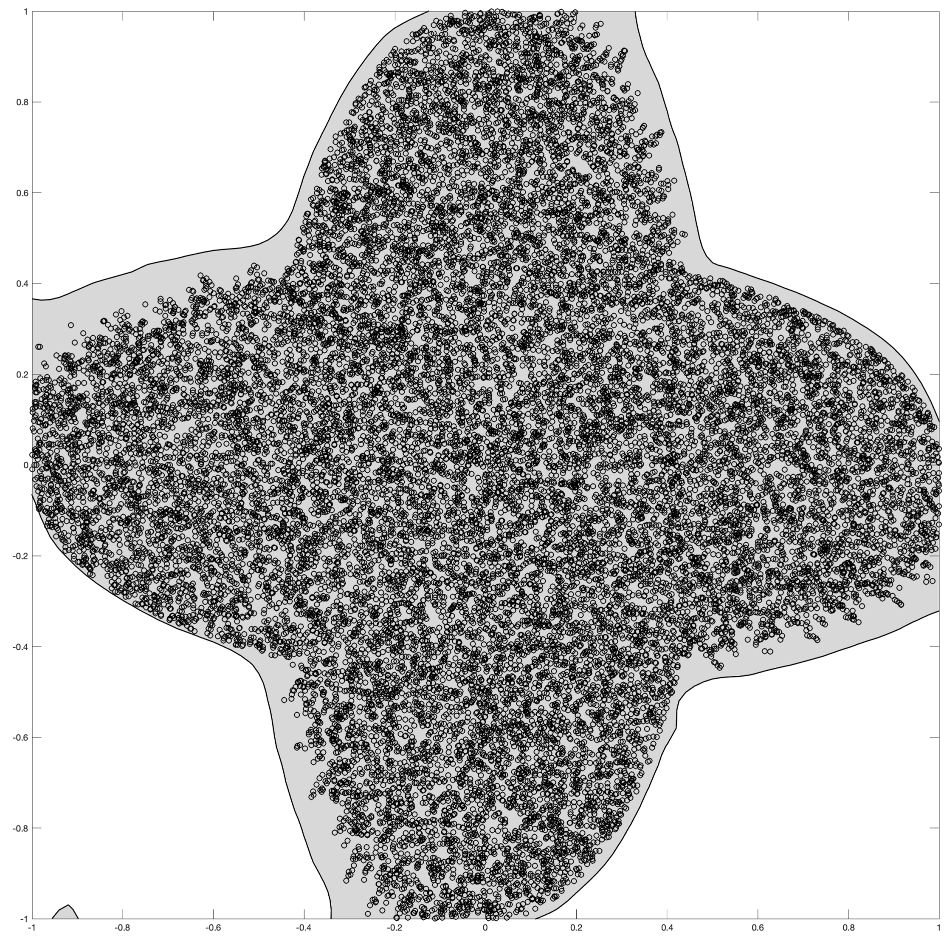


Switched system

Flower system
$$\begin{cases} \dot{x} = A_1 x, & x \in \mathcal{X}_1 \\ \dot{x} = A_2 x, & x \in \mathcal{X}_2 \end{cases}$$

Basis: 400 RBFs

 # Samples: 10000



Switched system

Modified flower system

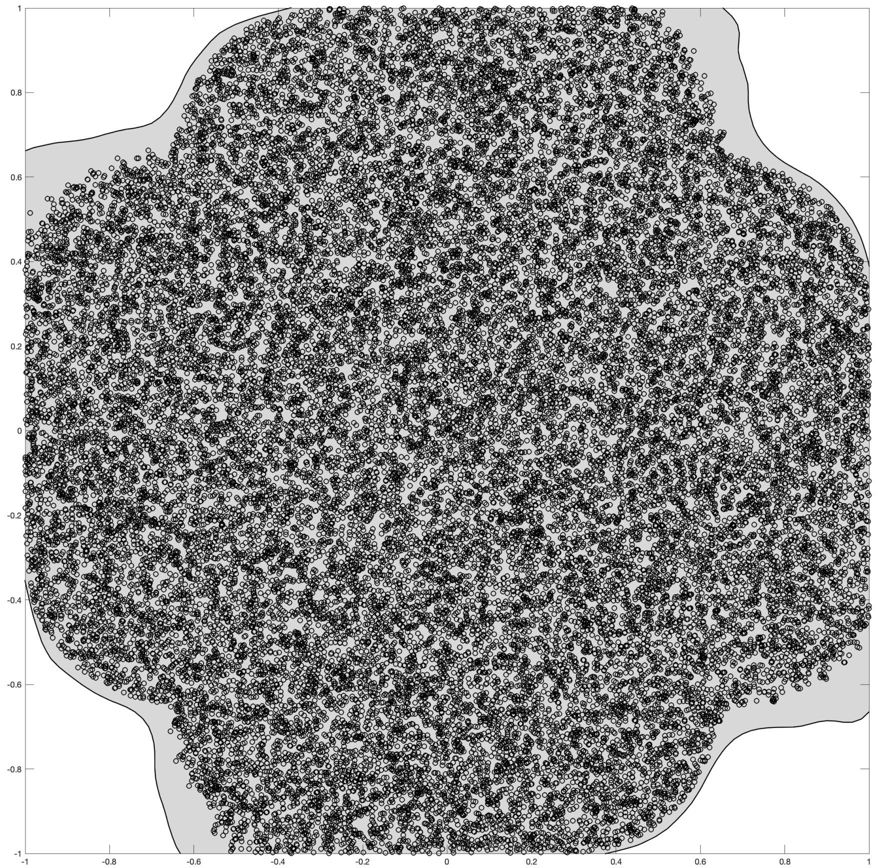
$$\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$$

Switched system

Modified flower system
$$\begin{cases} \dot{x} = A_1 \sin(x^3), & x \in \mathcal{X}_1 \\ \dot{x} = A_2 \sin(x^3), & x \in \mathcal{X}_2 \end{cases}$$

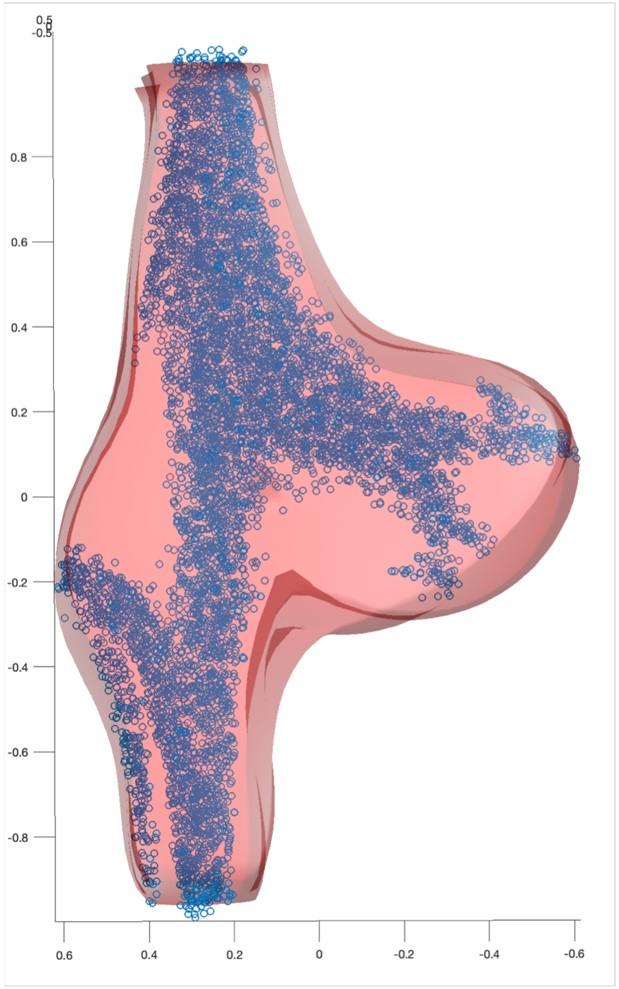
Basis: 400 RBFs

 # Samples: 10000

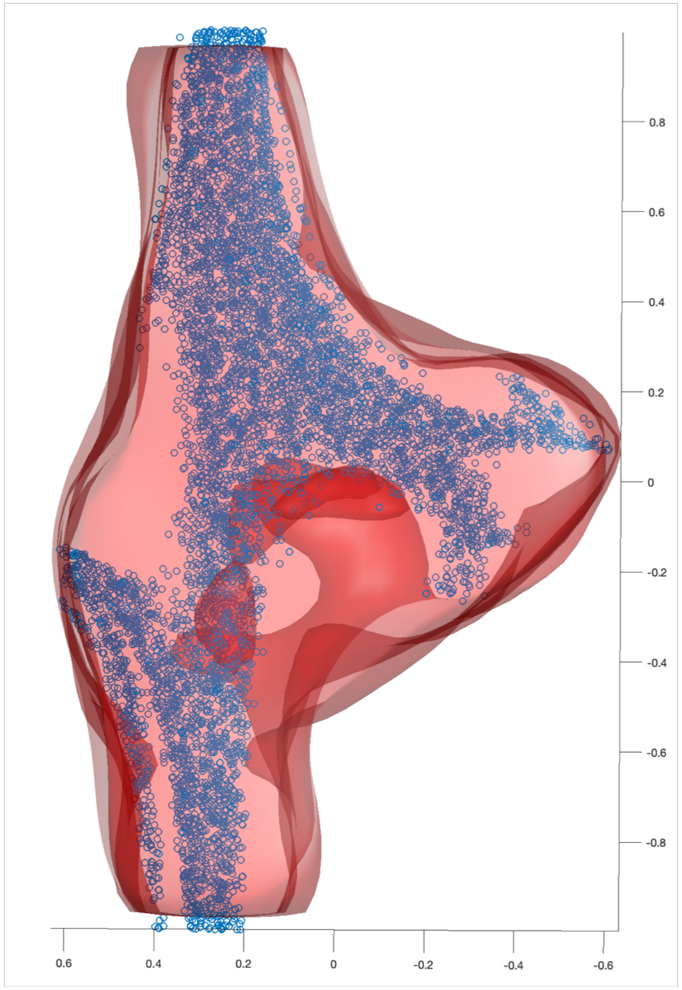


3D Hénon map

Basis: Monomials up to degree 10



Basis: 286 RBFs



Dimensionality dependence

$$f = [\underbrace{f_{\text{Julia}}, \ldots, f_{\text{Julia}}}_{n/2 \text{ times}}]^\top \qquad \Rightarrow \text{state-space of dimension } n$$

Box constraints: $-1 \le x_i \le 1$, $i = 1, \ldots, n$

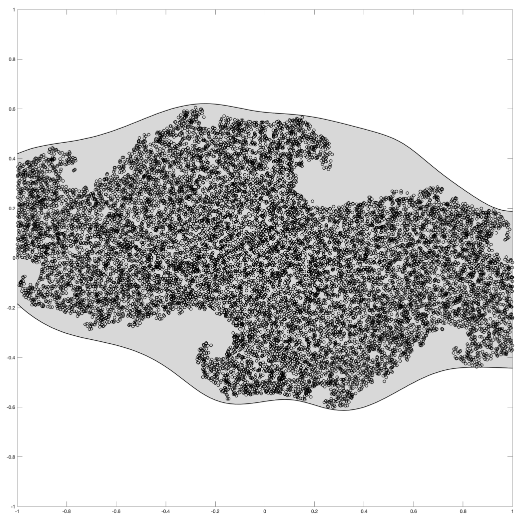
Random n-dimensional unitary state-space transformation

1600 thin-plate spline RBFs

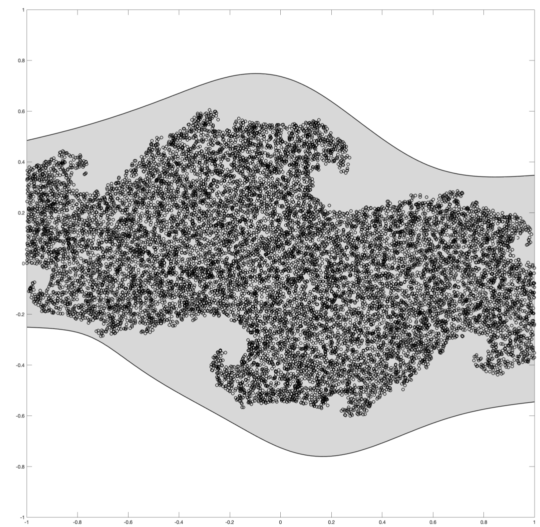
$3 \cdot 10^4$ samples

Dimensionality dependence

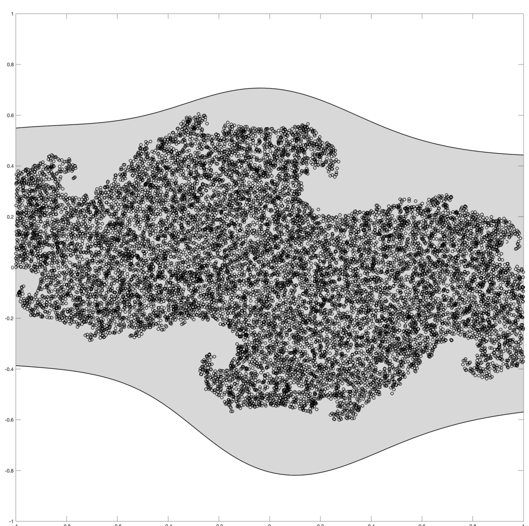
State-space dim = 4



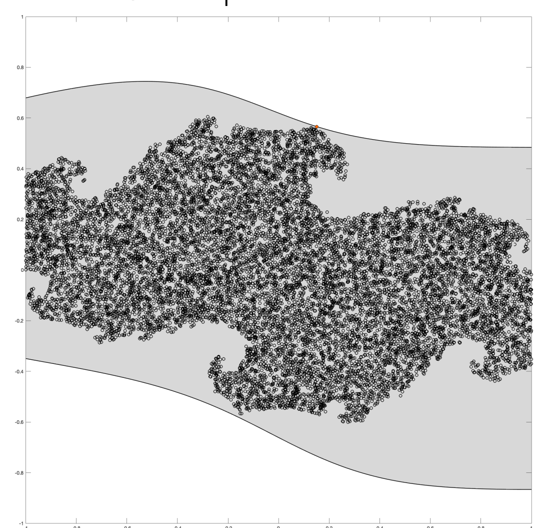
State-space dim = 6



State-space dim = 8



State-space dim = 10



Future work

More precise bounds (tightness?)

Iterative refining

Smooth distance and projection functions (more regularity on $v^\star$)

Iterated Bellman inequalities

Thank you