Set-based Co-Simulation Reachability Analysis

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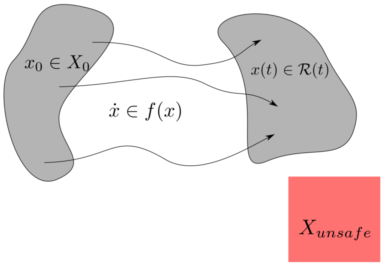
June 20, 2019

A dynamical system:

$$\begin{cases} \dot{x} \in f(x) \\ x(0) = x_0 \in X_0 \end{cases} \quad (1)$$

Reachable tube: $\mathcal{R}(t) = \{x(t) | x : [0, t] \mapsto \mathbb{R}^n \text{ satisfies Eq.(1)}\}$

Verify that each trajectory avoid X_{unsafe}.



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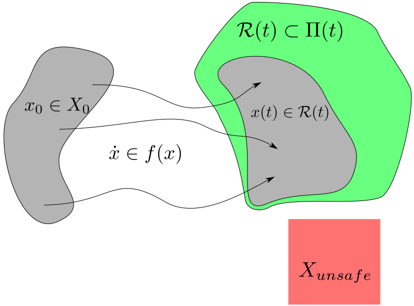

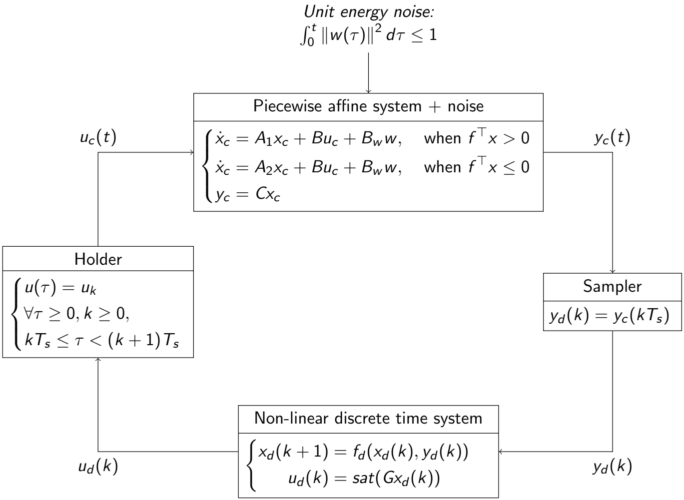
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$\int_0^t \|w(\tau)\|^2 \, d\tau \leq 1$
Piecewise affine system + noise
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$y_c(t)$
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$\begin{cases} u(\tau) = u_k \\ \forall \tau \geq 0, k \geq 0, \\ k T_s \leq \tau < (k+1) T_s \end{cases}$
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Non-linear discrete time system
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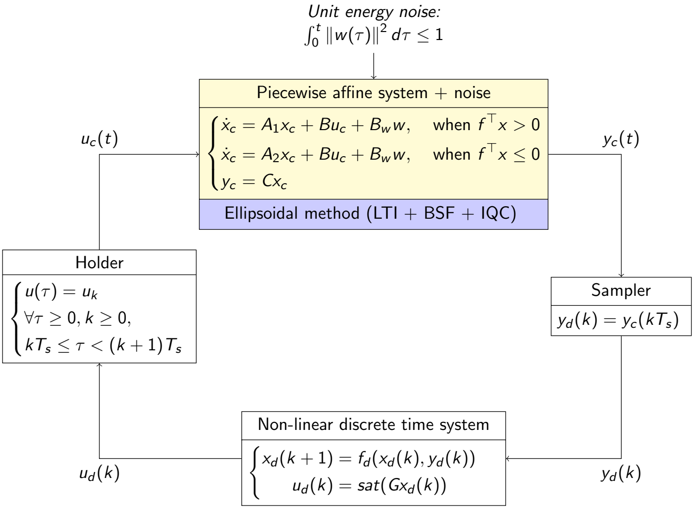
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Linear Time-Invariant system (LTI) + bounded noise:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ \|w(t)\|_{R(t)} \leq 1 \\ x(0) \in \mathcal{E}_{x0} \end{cases}$$

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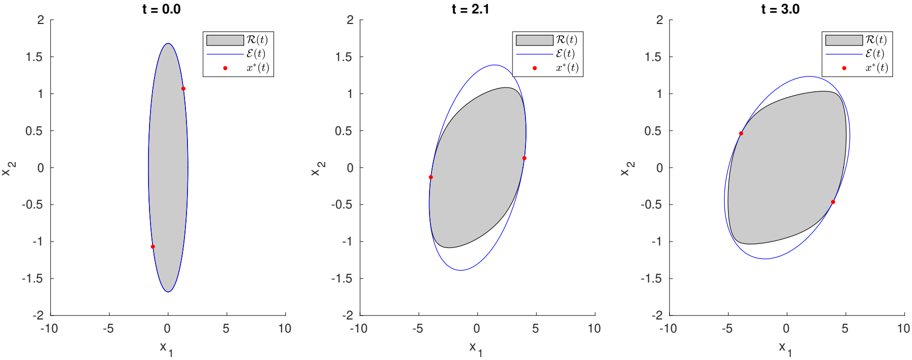

LTI system + bounded feedback:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ \|w(t)\|_{R(t)} \leq \|x(t) - x_c(t)\|_{Q(t)} \\ x(0) \in \mathcal{E}_{x0} \end{cases} \quad \Rightarrow \quad \mathcal{R}(t) \subset \mathcal{E}_x(t)$$

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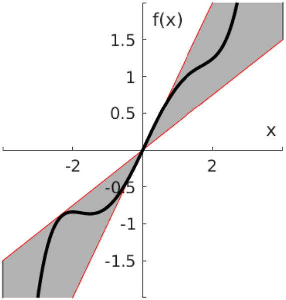

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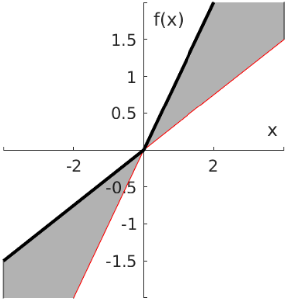

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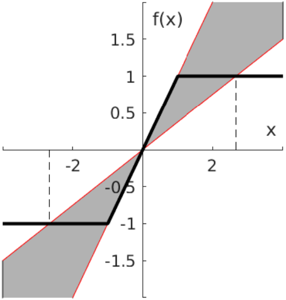
- Saturation



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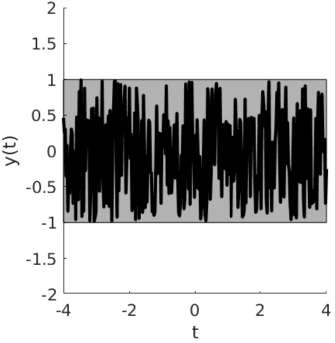
- Physical noise model



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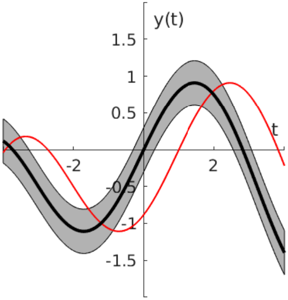

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- Physical noise model
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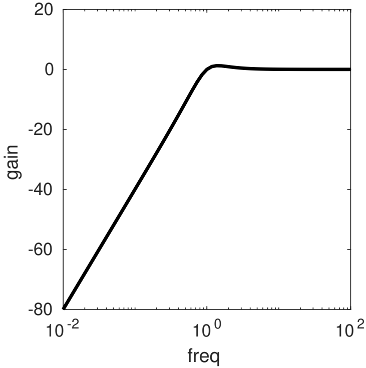
- Frequency observer

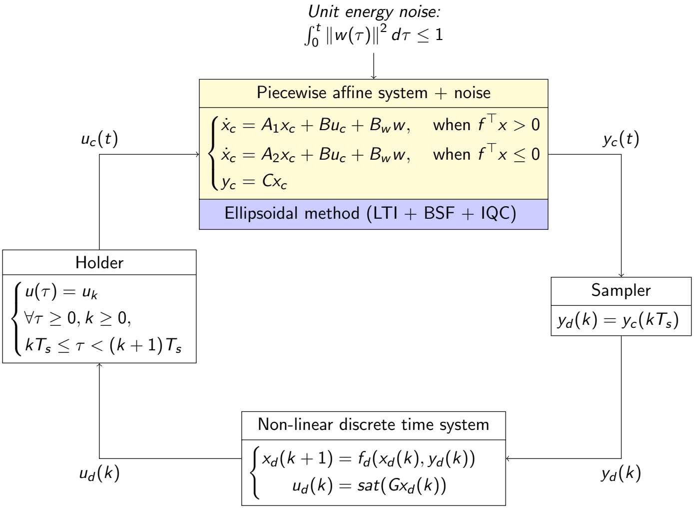


LTI system + Mixed disturbances:

Previous models can be combined:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w_1(t) + B_2 w_2(t) + \dots \\ \int_0^t \|w_i(\tau)\| d\tau \leq \int_0^t \|x(\tau) - x_c(\tau)\|_{Qi} d\tau + 1 \\ \|w_i(t)\| \leq \|x(t) - x_c(t)\|_{Qi(t)} + 1 \\ x(0) \in \mathcal{E}_{x0} \end{cases} \Rightarrow \mathcal{R}(t) \subset \mathcal{E}_x(t)$$

- Piecewise-Affine Linear systems
- Sector bounded non-linearities
- Delay systems
- Bounded energy disturbance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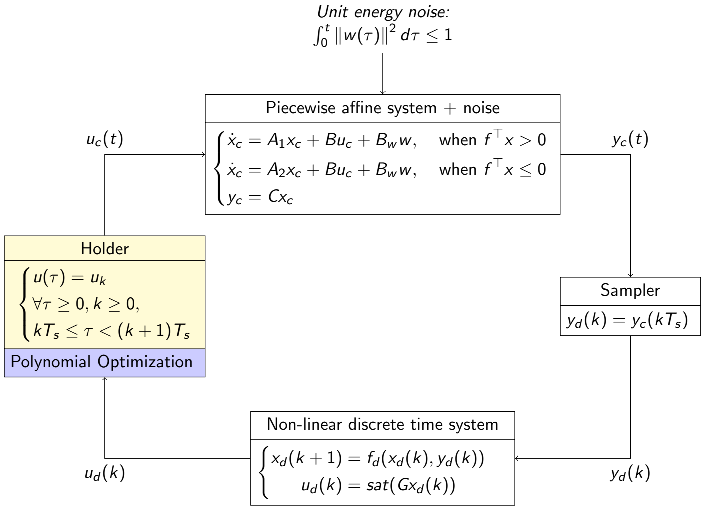
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We compute:

$$U(t) = \{u | u \in [u_{inf}(t), u_{sup}(t)]\}$$

with u_{inf} et u_{sup}:

$$\begin{array}{rl} \text{minimize} & \int_0^T u_{sup}(t)dt \\ \text{subject to} & u_{sup} \in \mathbb{R}_6[t] \\ & kT_s \leq t < (k+1)Ts, \\ & u_{sup}(t) \geq u(k) \end{array}$$

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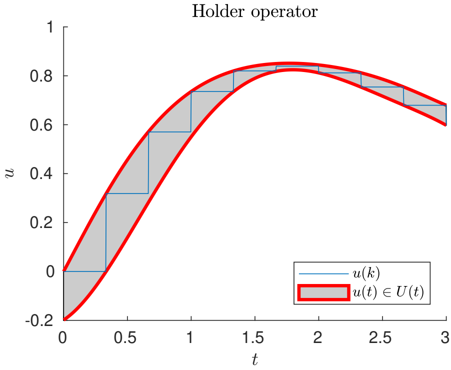

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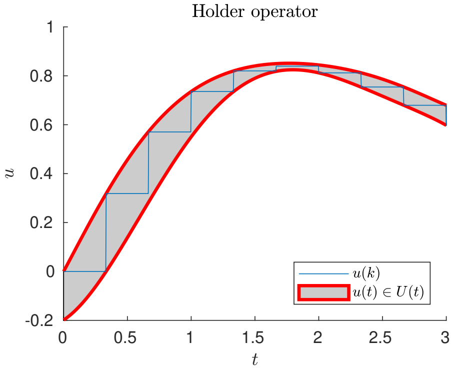
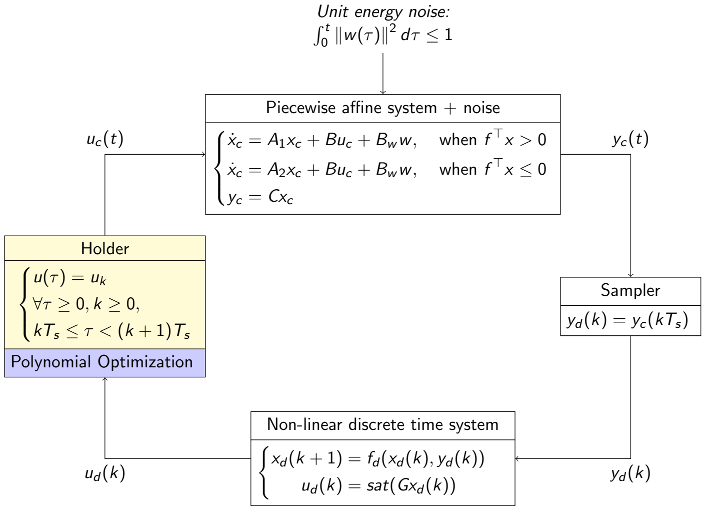


Solved with Sum-Of-Square programming.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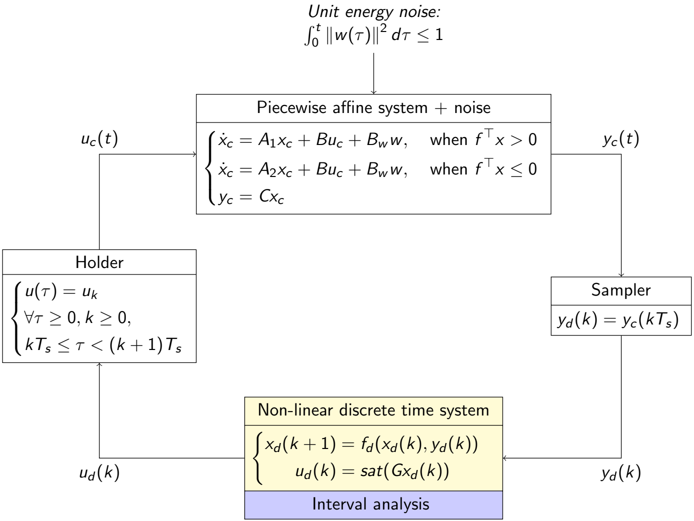
Some reachability analysis frameworks



- $\mathbb{IR}$ the set of intervals over $\mathbb{R}$:

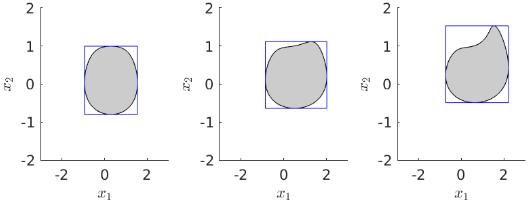
$$[\mathbf{x}] \in \mathbb{IR}^n$$

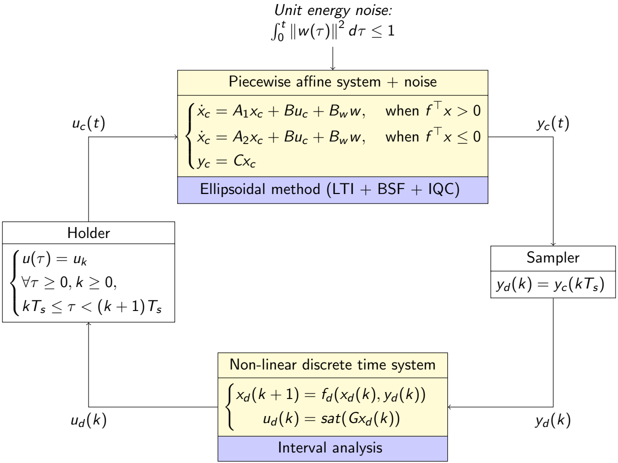
- each elementary operation $(+, -, *, /, \text{etc.})$ is replaced with its interval valuation:

$$\begin{aligned} [-2, 5] + [-8, 12] &= [-10, 17] \\ \left[\frac{3}{12}, \frac{5}{8}\right] \times [8, 12] &= \left[2, \frac{15}{2}\right] \end{aligned}$$

- Overapproximation of the reachable set:

$$[\mathbf{x}_d^{k+1}] = [f_d]\ ([\mathbf{x}_d^k])$$

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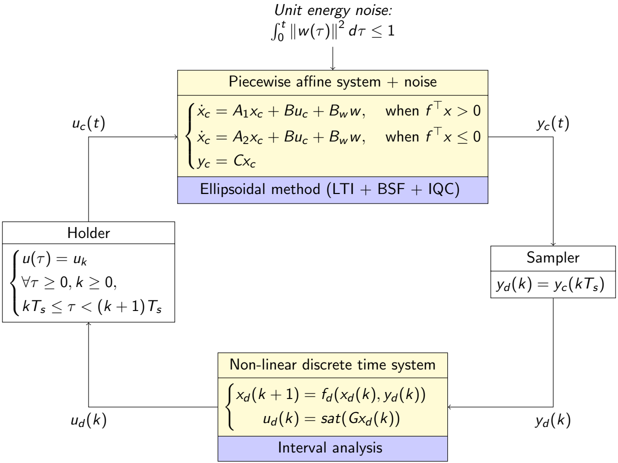
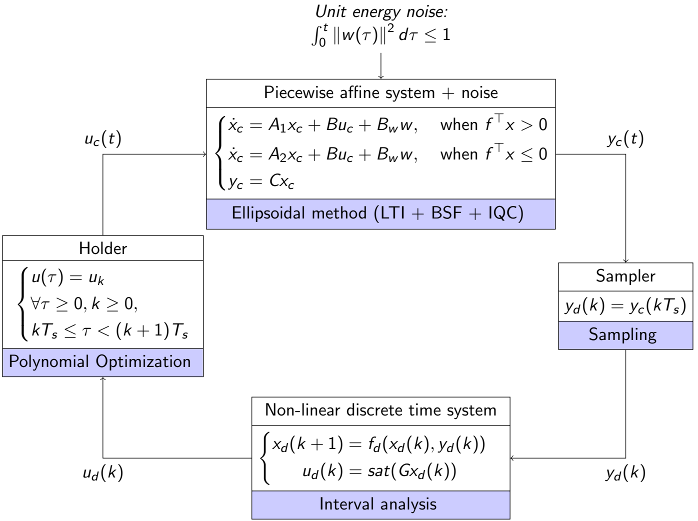


$\Rightarrow$ which framework should we choose?

Fixed Point Algorithm



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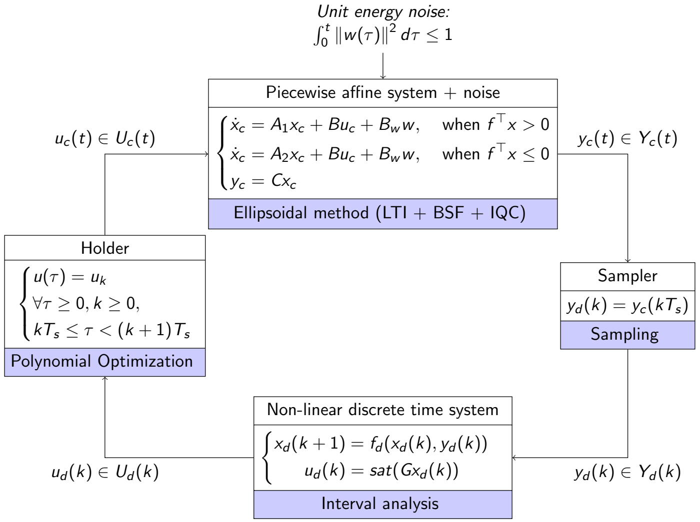

We overapproximate the reachable tube $t \to \mathcal{R}(t)$ of the system by constructing a sequence of sets $Z = (X_c, X_d, U_d, U_c)$ defined over the time domain $[0, 1]$. Since $u_d(t)$ is the image of a saturation operator, $U_d(t) \subseteq [-1, 1]$ for $t \in [0, 1]$.

- Initialization $Z^0 = (X_c^0, X_d^0, U_d^0, U_c^0)$:

$$\begin{cases} U_d^0 = [0,1] \times [-1,1] \\ U_c^0 = \text{SOS}\,(U_d^0) \\ X_c^0 = \text{EM}\,(U_c^0) \\ Y_d^0 = \text{sampling}\,(Y_c^0) \end{cases}$$

- Fixed Point algorithm $Z^{k+1} = \mathcal{C}(Z^k)$:

$$\begin{cases} U_c^{k+1} = \text{SOS}\,(U_d^k) & \cap \quad U_c^k \\ X_c^{k+1} = \text{EM}\,(U_c^k) & \cap \quad X_c^k \\ Y_d^{k+1} = \text{sampling}\,(Y_c^k) & \cap \quad Y_d^k \\ X_d^{k+1} = \text{IA}\,(U_d^k) & \cap \quad X_d^k \end{cases}$$

$\mathcal{C}$ is **monotonic**:

$$\mathcal{C}(Z) \subseteq Z$$

$\Rightarrow$ the sequence $\{Z^k\}_{k\in\mathbb{N}}$ is decreasing:

$$\emptyset \subseteq \cdots \subseteq Z^{k+1} \subseteq Z^k \subseteq \cdots \subseteq Z^1 \subseteq Z^0$$

and each k^{th} iterate Z^k overapproximates the reachable tube $\mathcal{R}(t)$, i.e. $\forall t \in [0,1]$:

$$\mathcal{R}(t) \subseteq Z^k(t).$$

Definition of the contraction operator $Z^{k+1} = \mathcal{C}(Z^k)$:

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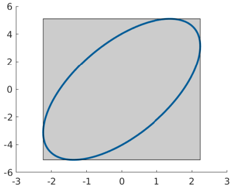
Overapproximate an ellipsoid $\mathcal{E}$ with box $\mathcal{B} = E2B(\mathcal{E})$:



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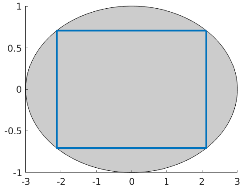

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Intersection's overapproximation:

$$\begin{aligned} \text{minimize } & \|\mathcal{E}\| \\ \text{s.t. } & \mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \mathcal{E} \end{aligned}$$

where $\|\mathcal{E}\| = \mathit{trace}(A)$ when

$$\mathcal{E} = \{x | (x - x_c)^\top A^{-1} (x - x_c) \leq 1\}$$

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- Ellipsoids: the intersection is not an ellipsoid!

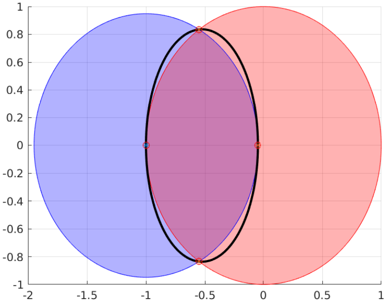
Intersection's overapproximation:

$$\text{minimize } \|\mathcal{E}\| \quad \text{s.t. } \mathcal{E}_1 \cap \mathcal{E}_2 \subseteq \mathcal{E}$$

where $\|\mathcal{E}\| = trace(A)$ when

$$\mathcal{E} = \{x | (x - x_c)^\top A^{-1} (x - x_c) \leq 1\}$$

$\Rightarrow$ solved using an SDP solver



Previous fixed point algorithm:

$$\mathcal{C}(Z) \subseteq Z$$

ixed point algorithm:

$$\widetilde{\mathcal{C}}(Z) \,\widetilde{\subseteq}\, Z \quad \Leftrightarrow \quad \begin{cases} \forall k \in \{0, \ldots, \frac{1}{T_s}\}, \mathcal{B}^+(k) \subseteq \mathcal{B}(k) \\ \qquad \forall t \in [0,1], \|\mathcal{E}^+(t)\| \leq \|\mathcal{E}(t)\| \end{cases}$$

where

$$\begin{cases} Z = (\mathcal{B}, \mathcal{E}) \\ Z^+ = \widetilde{\mathcal{C}}(Z) = (\mathcal{B}^+, \mathcal{E}^+) \end{cases}$$

Previous fixed point algorithm:

$$\mathcal{C}(Z) \subseteq Z$$

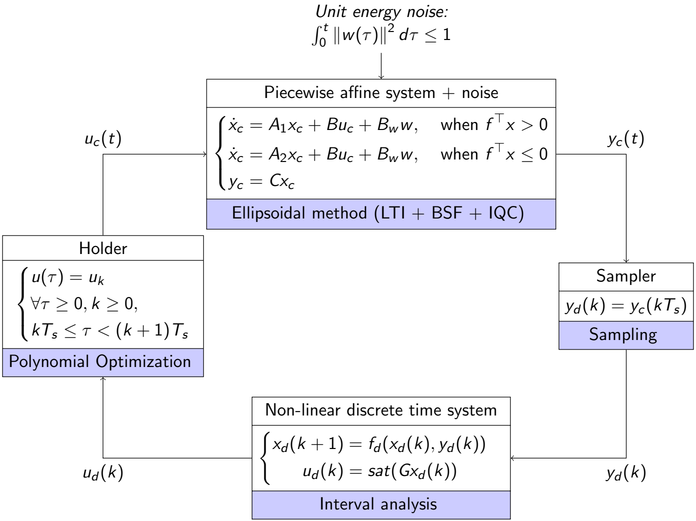
ixed point algorithm:

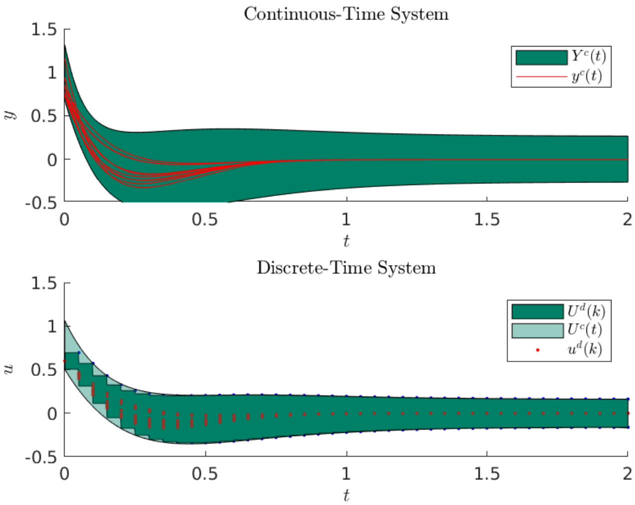
$$\widetilde{\mathcal{C}}(Z) \,\widetilde{\subseteq}\, Z \quad \Leftrightarrow \quad \begin{cases} \forall k \in \{0, \ldots, \frac{1}{T_s}\}, \mathcal{B}^{+}(k) \subseteq \mathcal{B}(k) \\ \forall t \in [0, 1], \|\mathcal{E}^{+}(t)\| \leq \|\mathcal{E}(t)\| \end{cases}$$

where

$$\begin{cases} Z = (\mathcal{B}, \mathcal{E}) \\ Z^{+} = \widetilde{\mathcal{C}}(Z) = (\mathcal{B}^{+}, \mathcal{E}^{+}) \end{cases}$$

Then, there exists a fixed point.


Unit energy noise:
$\int_0^t \|w(\tau)\|^2 \, d\tau \leq 1$
Piecewise affine system + noise
$\begin{cases} \dot{x}_c = A_1 x_c + B u_c + B_w w, & \text{when } f^\top x > 0 \\ \dot{x}_c = A_2 x_c + B u_c + B_w w, & \text{when } f^\top x \leq 0 \\ y_c = C x_c \end{cases}$
Ellipsoidal method (LTI + BSF + IQC)
$u_c(t)$
$y_c(t)$
Holder
$\begin{cases} u(\tau) = u_k \\ \forall \tau \geq 0, k \geq 0, \\ kT_s \leq \tau < (k+1)T_s \end{cases}$
Polynomial Optimization
Sampler
$y_d(k) = y_c(kT_s)$
Sampling
Non-linear discrete time system
$\begin{cases} x_d(k+1) = f_d(x_d(k), y_d(k)) \\ u_d(k) = sat(Gx_d(k)) \end{cases}$
Interval analysis
$u_d(k)$
$y_d(k)$


Continuous-Time System
1.5
1
0.5
0
-0.5
y
0
0.5
1
1.5
2
t
$Y^c(t)$
$y^c(t)$
Discrete-Time System
1.5
1
0.5
0
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u
0
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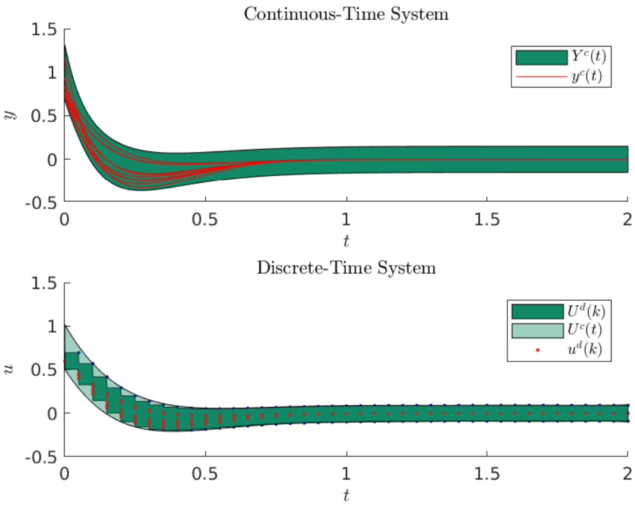
1
1.5
2
t
$U^d(k)$
$U^c(t)$
$u^d(k)$


Continuous-Time System
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y
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$Y^c(t)$
$y^c(t)$
Discrete-Time System
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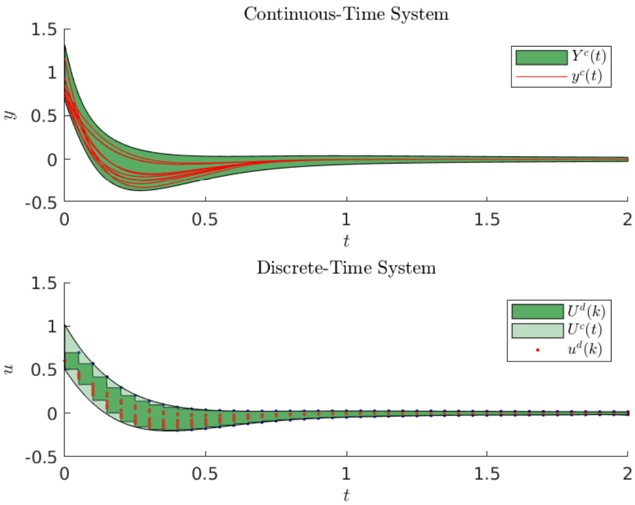
t
$U^d(k)$
$U^c(t)$
$u^d(k)$



Continuous-Time System
$Y^c(t)$
$y^c(t)$
y
t
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
Discrete-Time System
$U^d(k)$
$U^c(t)$
$u^d(k)$
u
t
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2

Advantages:

- we use a specialized model/methods for each subsystem;
- each iteration gives a valid overapproximation of the reachable tube.

Future works:

- Try to interface it with more set-based simulation frameworks;
- right now, the trajectories of each subsystem is independant form other subsystems. It implies a lot of conservatisme. How can we improve?