Analysis, Design, and Model Reduction of Consensus Networked Dynamical Systems

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References

- O. Farhat, **D. Abou Jaoude**, and M. Hudoba de Badyn, "H_∞ network optimization for edge consensus," European Journal of Control, Volume 62, 2021, Pages 2–13.
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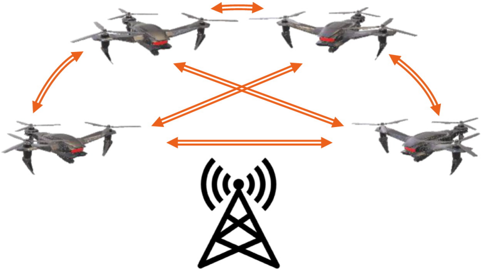
COOL

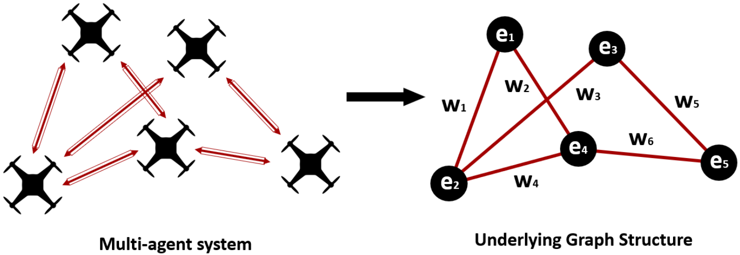
Control and Optimization Lab

- Dynamical systems operating over networks appear in many natural and engineering systems.
- Example applications include: robotics and autonomous spacecraft, wind farm optimization, and multi-agent systems.



Introduction

- A popular model of such dynamic processes is consensus, which is a distributed information-sharing protocol over a network where agents are able to agree on a common value of interest.



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  - For a connected graph, we can rely on the edge consensus model to perform the $\mathcal{H}_\infty$ analysis.

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- The state matrix of the consensus protocol is not Hurwitz, which precludes analysis involving the $\mathcal{H}_\infty$-norm.
  - For a connected graph, we can rely on the edge consensus model to perform the $\mathcal{H}_\infty$ analysis.

- We consider a network of single integrator agents operating on independent time scales, connected by weighted edges, and corrupted by exogenous disturbances.

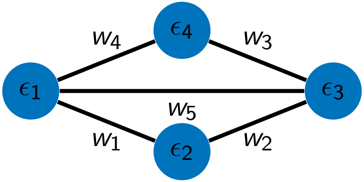
Relevant Literature

Relevant Prior Work	
(Zelazo & Mesbahi, 2011)[1]	• Developed the edge consensus protocol • Examined $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance
(Foight et al., 2020)[2]	• Incorporated edge weights and time scales • Examined $\mathcal{H}_2$ performance • Formulated $\mathcal{H}_2$ minimization problems

[1] D. Zelazo, M. Mesbahi, Edge agreement: Graph-theoretic Performance Bounds and Passivity Analysis, IEEE Transactions on Automatic Control 56 (3) (2011) 544-555.

[2] D. R. Foight, M. Hudoba de Badyn, M. Mesbahi, Performance and Design of Consensus on Matrix-Weighted and Time Scaled Graphs, IEEE Transactions on Control of Network Systems 7 (4) (2020) 1812-1822.

- Consider a network of n single integrator units evolving at differing rates.
- This configuration is represented by an **undirected** and **connected** graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W, E)$, where
  - $\mathcal{V}$: set of nodes,
  - $\mathcal{E}$: set of edges,
  - W: diagonal matrix of (positive) edge weights,
  - E: diagonal matrix of (positive) node time scales.



$$\mathcal{V} = \{1, 2, 3, 4\}, \ \mathcal{E} = \{12, 23, 34, 41, 13\},$$

$$W = \begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & w_4 & 0 \\ 0 & 0 & 0 & 0 & w_5 \end{bmatrix},$$

$$E = \begin{bmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{bmatrix}.$$

- For the graph $\mathcal{G}$:

1. The incidence matrix, which characterizes the incidence relation between distinct pairs of nodes, is

$$D(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}.$$

2. We also define the matrix $R(\mathcal{G}) = \begin{bmatrix} I & T_\tau^c \end{bmatrix}$, with

$$T_\tau^c = (D_\tau^T D_\tau)^{-1} D_\tau^T D_c,$$

where

- D_τ: incidence matrix of chosen spanning tree subgraph,
- D_c: incidence matrix of corresponding co-tree.

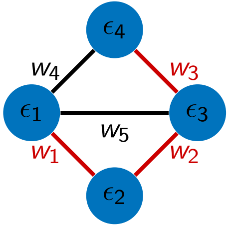


Figure: Red edges are the chosen **spanning tree** edges. Black edges are the corresponding **co-tree** edges.

Edge Consensus

- For a given spanning tree $\mathcal{G}_\tau$, the *edge consensus* model Σ_τ corresponding to the spanning tree edge states is given by

Edge Agreement Protocol (Foight et al., 2020)

$$\begin{cases} \dot{x}_\tau(t) = -L^\tau_{e,s} RWR^T x_\tau(t) + D^T_\tau E^{-1}\Omega\hat{w}(t) - L^\tau_{e,s} R\Gamma\hat{v}(t), \\ z(t) = R^T x_\tau(t), \end{cases}$$

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- $x_\tau(t) \in \mathbb{R}^{n-1}$: vector of edge states associated with the spanning tree edges,
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- $z(t)$: monitored performance signal.

$\mathcal{H}_\infty$ Performance

Edge Agreement Protocol Σ_τ

$$\begin{cases} \dot{x}_\tau(t) = \underbrace{-L_{e,s}^\tau RWR^T}_{A} x_\tau(t) + \underbrace{\begin{bmatrix} D_\tau^T E^{-1}\Omega & -L_{e,s}^\tau R\Gamma \end{bmatrix}}_{B} \begin{bmatrix} \hat{w}(t) \\ \hat{v}(t) \end{bmatrix}, \\ z(t) = \underbrace{R^T}_{C} x_\tau(t). \end{cases}$$

Then, $\Sigma_\tau(s) = R^T(sI + L_{e,s}^\tau RWR^T)^{-1} \begin{bmatrix} D_\tau^T E^{-1}\Omega & -L_{e,s}^\tau R\Gamma \end{bmatrix}$.

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- The $\mathcal{H}_\infty$-norm of a stable LTI system with a transfer function $\Phi(s)$ is defined as

$$\|\Phi\|_\infty = \sup_{\omega\in\mathbb{R}} \{\bar{\sigma}\big(\Phi(j\omega)\big)\}.$$

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Lemma (Zelazo & Mesbahi, 2011)

The $\mathcal{H}_\infty$-norm of the system Σ_τ satisfies $\|\Sigma_\tau\|_\infty = \bar{\sigma}\big(\Sigma_\tau(0)\big)$.

$\mathcal{H}_\infty$ Performance

Lemma

The $\mathcal{H}_\infty$-norm of the system Σ_τ satisfies

$$\|\Sigma_\tau\|_\infty^2 \geq \big(\lambda_{\min}(Q)\lambda_{\min}(B_\tau^T B_\tau) + \lambda_{\min}(F)\lambda_{\min}(B_c^T B_c)\big)\lambda_{\max}(J),$$
$$\|\Sigma_\tau\|_\infty^2 \leq \big(\lambda_{\max}(Q)\lambda_{\max}(B_\tau^T B_\tau) + \lambda_{\max}(F)\lambda_{\max}(B_c^T B_c)\big)\lambda_{\max}(J),$$

where $B_\tau = E^{-1}D_\tau$, $B_c = R^T L_{e,s}^\tau$, $Q = \Omega\Omega^T$, $F = \Gamma\Gamma^T$, and $J = A^{-T}C^T CA^{-1}$.

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- Hence, the $\mathcal{H}_\infty$ performance of two systems having pairs of covariance matrices that share the same maximum and minimum eigenvalues will be governed by the same bounds.
- The remainder of the results focus on the special choice of covariance matrices $\Omega = \sigma_w E^{\frac{1}{2}}$ and $\Gamma = \sigma_v W^{\frac{1}{2}}$, and the resulting system will be denoted by $\tilde{\Sigma}_\tau$.

$\mathcal{H}_\infty$ Performance

Theorem

The $\mathcal{H}_\infty$-norm of the system $\tilde{\Sigma}_\tau$ satisfies

$$\|\tilde{\Sigma}_\tau\|_\infty^2 = \bar{\sigma}(Z),$$

where

$$Z = \sigma_w^2 R^T (RWR^T L_{e,s}^\tau RWR^T)^{-1} R + \sigma_v^2 R^T (RWR^T)^{-1} R,$$

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- For $W = I$ and $E = I$ (Zelazo & Mesbahi, 2011):

$$\|\tilde{\Sigma}_\tau\|_\infty^2 = \bar{\sigma}\left(\sigma_w^2 R^T (RR^T D_\tau^T D_\tau RR^T)^{-1} R + \sigma_v^2 R^T (RR^T)^{-1} R\right)$$

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$$= \sigma_w^2 \bar{\sigma}(R^T (RR^T L_e^\tau RR^T)^{-1} R) + \sigma_v^2.$$

$\mathcal{H}_\infty$ Performance

- Consider a modified system Π_τ defined by

$$\Pi_\tau(s) = W^{\frac{1}{2}}\tilde{\Sigma}_\tau(s).$$

Theorem

The $\mathcal{H}_\infty$-norm of the system Π_τ satisfies

$$\|\Pi_\tau\|_\infty^2 = \sigma_w^2\bar{\sigma}(X) + \sigma_v^2,$$

where

$$X = W^{\frac{1}{2}}R^T(RWR^T L_{e,s}^\tau RWR^T)^{-1}RW^{\frac{1}{2}}.$$

$\mathcal{H}_\infty$ Performance

- The expression of $\|\Pi_\tau\|_\infty^2$ can be used to calculate new upper and lower bounds on the $\mathcal{H}_\infty$-norm of the original system $\tilde{\Sigma}_\tau$.

Theorem

The $\mathcal{H}_\infty$-norm of the system $\tilde{\Sigma}_\tau$ satisfies

$$\frac{\|\Pi_\tau\|_\infty}{\lambda_{\max}(W^{\frac{1}{2}})} \leq \|\tilde{\Sigma}_\tau\|_\infty \leq \frac{\|\Pi_\tau\|_\infty}{\lambda_{\min}(W^{\frac{1}{2}})}.$$

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- The upper bound to lower bound ratio η on $\|\tilde{\Sigma}_\tau\|_\infty$ is

$$\eta = \frac{\lambda_{\max}(W^{\frac{1}{2}})}{\lambda_{\min}(W^{\frac{1}{2}})}.$$

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- The upper bound to lower bound ratio η on $\|\tilde{\Sigma}_\tau\|_\infty$ is

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- If $W = I$ (unweighted graph):

$$\|\tilde{\Sigma}_\tau\|_\infty = \|\Pi_\tau\|_\infty,$$

and the obtained bounds are equal.

$\mathcal{H}_\infty$ Performance

Numerical Example

- To illustrate the obtained bounds, we consider the graph $\mathcal{G}$.
- The $\mathcal{H}_\infty$-norm of the corresponding system $\tilde{\Sigma}_\tau$ and the obtained bounds are computed for different combinations of edge weights and time scales.

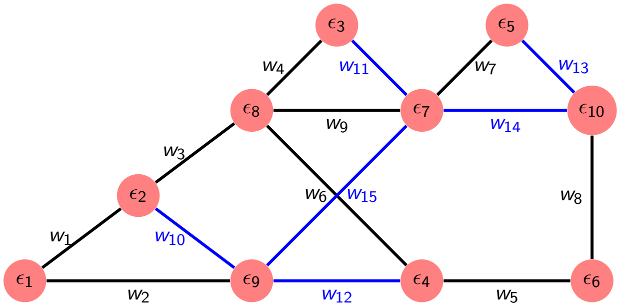

Figure: Black edges correspond to the chosen spanning tree edges. Blue edges correspond to the corresponding co-tree edges.

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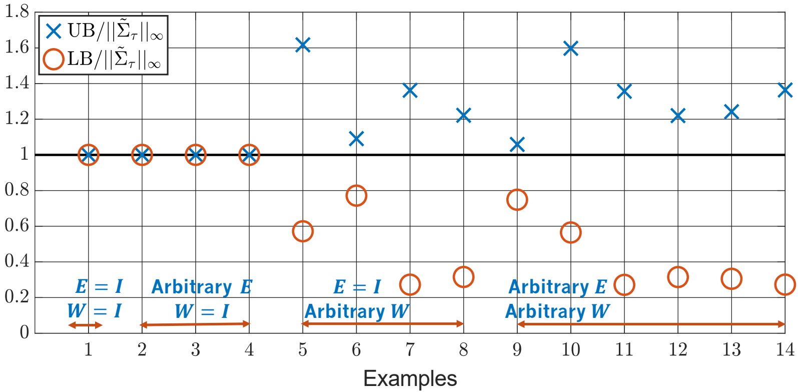

- We derive new insights on the $\mathcal{H}_\infty$-norm minimization problem.

Proposition

Consider the homogeneous bounds

$$w_{\min} I \preceq W \preceq w_{\max} I, \ \epsilon_{\min} I \preceq E \preceq \epsilon_{\max} I.$$

Then, $E = \epsilon_{\min} I$ and $W = w_{\max} I$ minimize $\|\Pi_\tau(E, W)\|_\infty$.
Hence,

$$\|\tilde{\Sigma}_\tau\|_\infty = \frac{\|\Pi_\tau\|_\infty}{\sqrt{w_{\max}}}$$

is also minimized.

Optimal Time Scales and Edge Weights

- We propose the following optimization paradigm if diversity of time scales and edge weights is desirable in the particular application of interest.

$$\min \|\Pi_\tau\|_\infty^2 = \sigma_w^2 \bar{\sigma}(X) + \sigma_v^2 \Leftrightarrow \min \lambda_{\max}(X) \Leftrightarrow \begin{array}{cc} \min & \zeta \\ s.t & X \preceq \zeta I \end{array},$$

where $X = W^{\frac{1}{2}} R^T (RWR^T L_{e,s}^{\tau} RWR^T)^{-1} RW^{\frac{1}{2}}$.

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where $X = W^{\frac{1}{2}} R^T (RWR^T L_{e,s}^\tau RWR^T)^{-1} RW^{\frac{1}{2}}$.

- To formulate our problem as a convex optimization problem, we minimize $\lambda_{\max}(X_1)$ instead, where

$$X_1 = W^{-\frac{1}{2}} R^\dagger (L_{e,s}^\tau)^{-1} (R^\dagger)^T W^{-\frac{1}{2}},$$

and

$$\lambda_{\max}(X_1) \geq \lambda_{\max}(X).$$

Optimal Time Scales and Edge Weights

- Minimization Problem:

$$\min_{\zeta, w_l^{-\frac{1}{2}}, \epsilon_i^{-1}} \quad \zeta$$
$$s.t \quad X_1 \preceq \zeta I.$$

- $X_1 \preceq \zeta I \Leftrightarrow \begin{bmatrix} \zeta I & W^{-\frac{1}{2}} R^{\dagger} \\ (W^{-\frac{1}{2}} R^{\dagger})^T & L_{e,s}^{\tau} \end{bmatrix} \succeq 0,$
  where $L_{e,s}^{\tau} = D_{\tau}^T E^{-1} D_{\tau}$.

  - ϵ_i: time scale associated with node i,
    $\epsilon^{-1} = (\epsilon_1^{-1}, \ldots, \epsilon_n^{-1})$ and $E^{-1} = \mathbf{diag}(\epsilon^{-1})$,

  - w_l: edge weight associated with edge l,
    $w^{-\frac{1}{2}} = (w_1^{-\frac{1}{2}}, \ldots, w_{|\mathcal{E}|}^{-\frac{1}{2}})$ and $W^{-\frac{1}{2}} = \mathbf{diag}(w^{-\frac{1}{2}})$.

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Optimal Time Scales and Edge Weights

1. To penalize small node time scales, a regularization term $\|\epsilon^{-1}\|_2$ is added to the objective function.

Optimal Time Scales and Edge Weights

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2. To penalize large edge weights, a new variable $\xi \in \mathbb{R}^{|\mathcal{E}|}$ is introduced such that $w_l^{\frac{1}{2}} \leq \xi_l \Leftrightarrow W^{\frac{1}{2}} \preceq \Xi$, with $\Xi = \textbf{diag}(\xi)$.
   Thus, a regularization term $\|\xi\|_2$ is added to the objective function and the LMI
   $$\begin{bmatrix} \Xi & I \\ I & W^{-\frac{1}{2}} \end{bmatrix} \succeq 0$$
   is added to the set of constraints.

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   $$\begin{bmatrix} \Xi & I \\ I & W^{-\frac{1}{2}} \end{bmatrix} \succeq 0$$
   is added to the set of constraints.

3. To tighten the bounds on the $\mathcal{H}_\infty$-norm of $\tilde{\Sigma}_\tau$, an upper bound γ is imposed on the ratio η, which can be done through adding the convex constraint
   $$\lambda_{\max}(W^{-\frac{1}{2}}) - \gamma\lambda_{\min}(W^{-\frac{1}{2}}) \leq 0.$$

Optimal Time Scales and Edge Weights

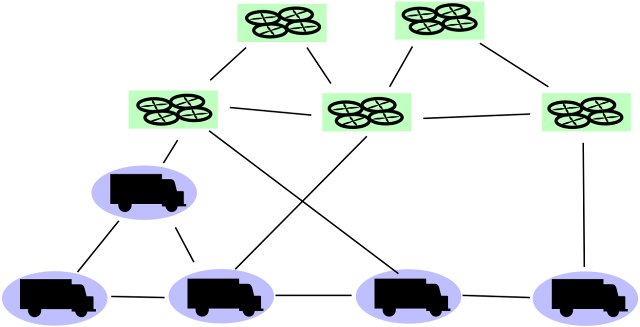
- We perform $\mathcal{H}_\infty$-norm minimization by solving the following semidefinite program:

$$
\begin{aligned}
\underset{\zeta, w_l^{-\frac{1}{2}}, \epsilon_i^{-1}, \xi}{\text{minimize}} \quad & \zeta + \alpha\|\xi\|_2 + \beta\|\epsilon^{-1}\|_2 \\
\text{subject to} \quad & \begin{bmatrix} \zeta I & W^{-\frac{1}{2}} R^{\dagger} \\ (R^{\dagger})^T W^{-\frac{1}{2}} & L_{e,s}^{\tau} \end{bmatrix} \succeq 0, \\
& w_{\max,l}^{-\frac{1}{2}} \leq w_l^{-\frac{1}{2}} \leq w_{\min,l}^{-\frac{1}{2}}, \quad \epsilon_{\max,i}^{-1} \leq \epsilon_i^{-1} \leq \epsilon_{\min,i}^{-1}, \\
& \begin{bmatrix} \Xi & I \\ I & W^{-\frac{1}{2}} \end{bmatrix} \succeq 0, \\
& \lambda_{\max}(W^{-\frac{1}{2}}) - \gamma \lambda_{\min}(W^{-\frac{1}{2}}) \leq 0,
\end{aligned}
$$

for all $i \in \mathcal{V}$ and $l \in \mathcal{E}$, where $\alpha > 0$ and $\beta > 0$ are weights on the different components of the objective function.

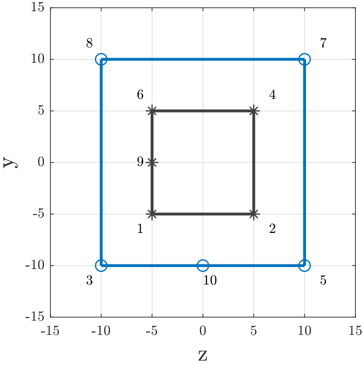
Formation Control Example

- Consider a network of non-homogenous agents of ground and aerial vehicles operating on multiple time scales.
- Faster agents (aerial vehicles) are in green squares.
- Slower agents (ground vehicles) are in blue ellipses.



Formation Control Example

- The agents are each assigned a position.
  - Slow agents: '$*$' markers on the border of the inner square.
  - Fast agents: '$\circ$' markers on the border of the outer square.
- The agents run a one-dimensional consensus protocol in two directions $\{z, y\}$.
- $t \in [2, 3]$ is a finite support of randomly generated disturbance signals on the nodes and edges.



- NUD: Unity edge weights and time scales
- BUD: Optimal edge weights and time scales

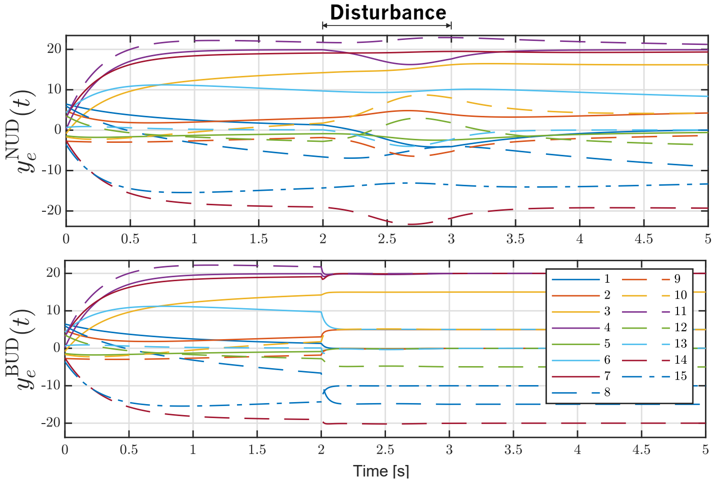


Figure: Edge states in the y direction over time.

Model Reduction of Consensus Network Systems

- As network systems increase in size and complexity, it becomes desirable to find lower order approximant network models.
- This part proposes $\mathcal{H}_\infty$- and $\mathcal{H}_2$-based model reduction (MR) methods for approximating the input-output behavior of a given consensus network system $\mathbf{\Sigma}$ (n agents) with a reduced consensus network system $\hat{\mathbf{\Sigma}}$ (r agents) based on a *predefined* clustering of the graph structure of $\mathbf{\Sigma}$.
- We consider consensus network systems (CNSs) consisting of **time-scaled single integrator** agents evolving over a network described by a **undirected**, **weighted**, and **connected** graph.

Prior Work and Paper Contributions

	Clustering-based MR approach		
Original CNS Σ	**Common approach**	**(Cheng et al., 2019)[1] (Undirected)** **(Cheng et al., 2020)[2] (Directed)**	**Our approach** (Undirected)
STEP 1: Choose clustering for Σ	Find a "good" clustering	Assume network clustering is given	
STEP 2: Obtain $\hat{\Sigma}$ based on clustering of Σ	Apply a projection operation	$\mathcal{H}_2$-based tuning of edge weights	$\mathcal{H}_2$/ $\mathcal{H}_\infty$-based tuning of edge weights $\mathcal{H}_2$/ $\mathcal{H}_\infty$-based tuning of nodal time-scales
Reduced CNS $\hat{\Sigma}$			

[1]X. Cheng, L. Yu, and J. Scherpen, "Reduced order modeling of linear consensus networks using weight assignments," in 18th European Control Conference (ECC), 2019, pp. 2005–2010.

[2]X. Cheng, L. Yu, D. Ren, and J. Scherpen, "Reduced order modeling of diffusively coupled network systems: An optimal edge weighting approach," arXiv preprint, arXiv:2003.03559, 2020.

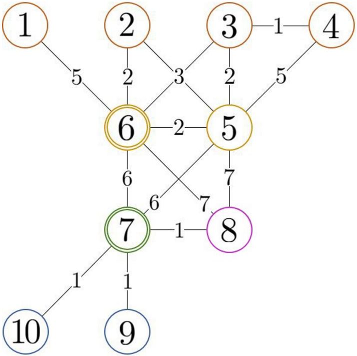
Problem Setup

Consensus Dynamics

$$\boldsymbol{\Sigma} : \begin{cases} E\dot{x}(t) = -Lx(t) + Fu(t), \\ y(t) = Hx(t), \end{cases}$$

where $L = DWD^T$ and D are the Laplacian and incidence matrices of $\mathcal{G}$.

- For example, consider the following consensus network $\mathbf{\Sigma}$:



- The configuration is represented by a graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, E, W)$ with
  - $\mathcal{V} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
  - $E = \textbf{diag}((1, 1, 1, 1, 1, 1, 1, 1, 1, 1))$
  - $\mathcal{E} = \{16, 25, 26, 34, 35, 36, 45, 56, 57, 58, 67, 68, 78, 79, 710\}$
  - $W = \textbf{diag}((5, 3, 2, 1, 2, 3, 5, 2, 6, 7, 6, 7, 1, 1, 1))$

Problem Setup

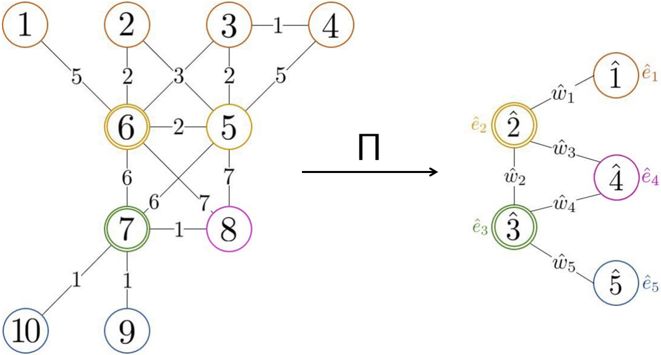
- The given graph clustering of $\mathcal{G}(\mathcal{V}, \mathcal{E}, E, W)$ can be characterized by the following matrix $\Pi \in \mathbb{R}^{10\times 5}$:

$$\Pi = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T .$$

- Using Π, a reduced graph $\hat{\mathcal{G}}$ is obtained as follows:
  - nodes within the same cluster are aggregated into a single node
  - edges within the same cluster are removed
  - if there is at least one edge between any pairs of nodes in different clusters, then a single edge between the corresponding clusters is retained. Otherwise, no edge exists between the two clusters.
- A reduced, **undirected**, and **connected** graph $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{E}, \hat{W})$ with arbitrary edge weights and nodal time-scales is obtained.

Problem Setup

- Thus, the following parameterized reduced consensus network system $\hat{\boldsymbol{\Sigma}}$ is constructed:



- The underlying reduced graph is given by $\hat{\mathcal{G}}(\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{E}, \hat{W})$ with
  - $\hat{\mathcal{V}} = \{1, 2, 3, 4, 5\}$
  - $\hat{E} = \mathbf{diag}((\hat{e}_1,\ \hat{e}_2,\ \hat{e}_3,\ \hat{e}_4,\ \hat{e}_5))$
  - $\hat{\mathcal{E}} = \{12, 23, 24, 34, 35\}$
  - $\hat{W} = \mathbf{diag}((\hat{w}_1,\ \hat{w}_2,\ \hat{w}_3,\ \hat{w}_4,\ \hat{w}_5))$

Problem Setup

Consensus Dynamics

$$\mathbf{\Sigma} : \begin{cases} E\dot{x}(t) = -DWD^T x(t) + Fu(t), \\ y(t) = Hx(t). \end{cases}$$

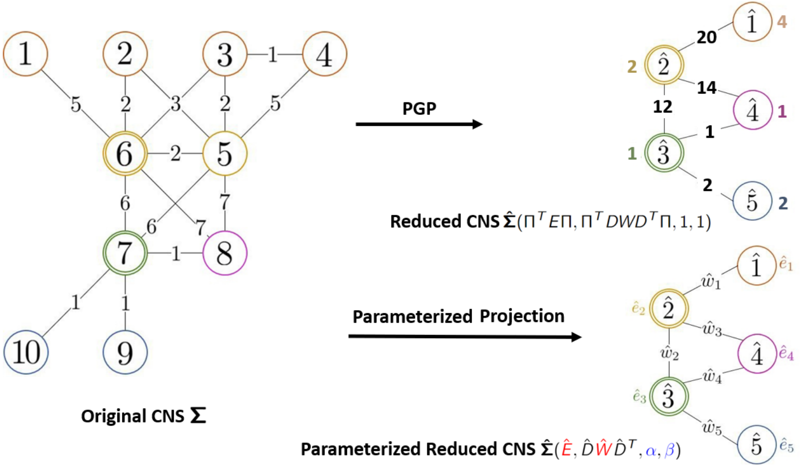
Reduced (Parameterized) Consensus Dynamics

$$\hat{\mathbf{\Sigma}}(\hat{E}, \hat{W}, \alpha, \beta) : \begin{cases} \hat{E}\dot{\hat{x}}(t) = -\hat{D}\hat{W}\hat{D}^T \hat{x}(t) + \beta\hat{F}u(t), \\ \hat{y}(t) = \alpha\hat{H}\hat{x}(t). \end{cases}$$

- Tune $\hat{E}$ and $\hat{W}$ so that $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty}$ or $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2}$ is minimized.
- $\hat{F} = \Pi^T F$, $\hat{H} = H\Pi$, and $\hat{D} = \Pi^T D$ (duplicate and zero columns removed).
- α and β are scalars chosen to ensure that any choice for $\hat{E}$ and $\hat{W}$ yields a Bounded-Input Bounded-Output (BIBO) stable error system.

Problem Setup

- The Petrov-Galerkin Projection (PGP) paradigm is often applied in clustering-based MR techniques after carefully choosing Π.
- Here, the reduced edge weights and nodal time-scales are parameterized and tuned for better results.



Problem Setup

Problem Statement

Given the consensus network system $\boldsymbol{\Sigma}$ and the parameterized reduced consensus network system $\hat{\boldsymbol{\Sigma}}$, solve the following optimization problems:

$$\min_{\hat{E}\in\mathbb{D}^r_{++},\hat{W}\in\mathbb{D}^{|\hat{\mathcal{E}}|}_{++}} \|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_\infty},$$

$$\min_{\hat{E}\in\mathbb{D}^r_{++},\hat{W}\in\mathbb{D}^{|\hat{\mathcal{E}}|}_{++}} \|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_2},$$

where $\mathbb{D}^n_{++}$ is the set of diagonal positive definite matrices of size $n\times n$.

(Cheng et al., 2019) and (Cheng et al., 2020)

$$\min_{\hat{W}\in\mathbb{D}^{|\hat{\mathcal{E}}|}_{++}} \|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_2} \text{ where } \hat{E}=\Pi^T E\Pi.$$

BIBO Stability of Error System

Realization for the error system $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$:

$$\mathbf{\Sigma} - \hat{\mathbf{\Sigma}} : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} -E^{-1}L & 0 \\ 0 & -\hat{E}^{-1}\hat{D}\hat{W}\hat{D}^T \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} + \begin{bmatrix} E^{-1}F \\ \beta\hat{E}^{-1}\hat{F} \end{bmatrix} u(t), \\ e(t) = \begin{bmatrix} H & -\alpha\hat{H} \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \end{cases}$$

Lemma 1

If $\alpha\beta = \frac{tr(\hat{E})}{tr(E)}$, then $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$ is BIBO stable for all $\hat{E} \in \mathbb{D}^r_{++}$ and $\hat{W} \in \mathbb{D}^{|\hat{\mathcal{E}}|}_{++}$.

BIBO Stability of Error System

A few remarks on Lemma 1:

- The condition on $\alpha\beta$ guarantees the BIBO stability of $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$ by cancelling the poles at 0 from the transfer function of $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$.
- For such parameters α and β, the approximation errors given by $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty}$ and $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2}$ are bounded for any $\hat{E}$ and $\hat{W}$.
- $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty}$ and $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2}$ are invariant under the specific choices of α and β as long as $\alpha\beta = \frac{tr(\hat{E})}{tr(E)}$.

Adopted convenient choice for α and β

$$\alpha = 1 \text{ and } \beta = \frac{tr(\hat{E})}{tr(E)}.$$

$\mathcal{H}_\infty$-based Optimization

- $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_\infty} < \hat{\gamma}$ can be characterized by a matrix inequality in $\hat{W}$ and $\hat{E}$:

$$\min_{\hat{\gamma}, X \succ 0, \hat{W}, \hat{E}} \quad \hat{\gamma}$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} A^T X + XA & XB & \theta C^T & XJ \\ * & -\hat{\gamma} I & 0 & 0 \\ * & * & -\hat{\gamma} I & 0 \\ * & * & * & 0 \end{bmatrix}}_{\psi_\infty(\hat{\gamma}, X)} + \underbrace{\begin{bmatrix} -\hat{A}^T \hat{A} & -\hat{A}^T \hat{B} & \theta \hat{C}^T & \hat{A}^T \\ * & -\hat{B}^T \hat{B} & 0 & \hat{B}^T \\ * & * & 0 & 0 \\ * & * & * & -I \end{bmatrix}}_{\phi_\infty(\hat{W}, \hat{E})} \prec 0.$$

- Although $\hat{W}$ and $\hat{E}$ are neatly decoupled from $\hat{\gamma}$ and X and $\psi_\infty(\hat{\gamma}, X)$ is linear in its arguments, $\phi_\infty(\hat{W}, \hat{E})$ is nonlinear in its arguments.
- We alternate between optimizing $\hat{W}$ for fixed $\hat{E}$ and vice-versa.

$\mathcal{H}_\infty$-based Optimization

$\mathcal{H}_\infty$-based optimization problem for fixed $\hat{E}$:

$$\min_{\hat{\gamma}, X \succ 0, \hat{W}} \quad \hat{\gamma}$$
$$\text{s.t.} \quad \underbrace{\psi_\infty(\hat{\gamma}, X)}_{\textbf{convex term}} + \underbrace{\phi_\infty(\hat{W})}_{\textbf{concave term}} \prec 0.$$

- Constraint is dealt with by:
  - Iteratively solving a *linearized* optimization problem, where the concave term is linearized around the previous solution $\hat{W}_{(k-1)}$.
  - Linearizing the concave term $\phi_\infty(\hat{W})$, while making the constraint more restrictive, makes the optimization problem convex.

Algorithm 1 (A1): $\mathcal{H}_\infty$-based optimization for fixed time-scales $\hat{E}$

Input: E, L, F, H, Π, $\hat{E}$, $\hat{W}_0$, $\hat{\gamma}_0$, ε

Output: $\hat{W}_*$

Initialize: $k \leftarrow 0$, $\hat{W}_{(k)} \leftarrow \hat{W}_0$, and $\hat{\gamma}_{(k)} \leftarrow \hat{\gamma}_0$

While: $\hat{\gamma}_{(k)} - \hat{\gamma}_{(k+1)} \geq \varepsilon$

Solve:

$$\min_{\hat{\gamma}, X \succ 0, \hat{W}} \quad \hat{\gamma}$$

$$\text{s.t.} \quad \psi_\infty(\hat{\gamma}, X) + \underbrace{\phi_\infty(\hat{W}_{(k-1)}) + D\phi_\infty(\hat{W}_{(k-1)})[\hat{W} - \hat{W}_{(k-1)}]}_{\tilde{\phi}_\infty(\hat{W})} \prec 0,$$

where $\phi_\infty(\hat{W}) \prec \tilde{\phi}_\infty(\hat{W}) \prec 0$.

Obtain: $\hat{W}_*$ and $\hat{\gamma}_*$

Update: $k \leftarrow k + 1$, $\hat{W}_{(k)} \leftarrow \hat{W}_*$, and $\hat{\gamma}_{(k)} \leftarrow \hat{\gamma}_*$ **end**

$\mathcal{H}_\infty$-based Optimization

$\mathcal{H}_\infty$-based optimization problem for fixed $\hat{W}$:

$$\min_{\hat{\gamma}, X \succ 0, \hat{E}} \quad \hat{\gamma}$$
$$\text{s.t.} \quad \underbrace{\psi_\infty(\hat{\gamma}, X)}_{\textbf{convex term}} + \underbrace{\phi_\infty(\hat{E})}_{\textbf{not concave}} \prec 0.$$

- In this case, $\phi_\infty(\hat{E})$ is *not* concave in $\hat{E}$.
- More involved manipulations are required to solve the problem.
- The structure of $\phi_\infty(\hat{E})$ can be leveraged by
  - Treating $\hat{E}^{-1}$ as the variable instead of $\hat{E}$.
  - Introducing a variable Z such that $Z = \hat{E}$.

$\mathcal{H}_\infty$-based Optimization

$\mathcal{H}_\infty$-based optimization problem for fixed $\hat{W}$:

$$\begin{aligned} \min_{\hat{\gamma}, X \succ 0, \hat{E}^{-1}, Z} \quad & \hat{\gamma} \\ \text{s.t.} \quad & \underbrace{\psi_\infty(\hat{\gamma}, X)}_{\textbf{convex term}} + \underbrace{\phi_\infty(\hat{E}^{-1}, Z)}_{\textbf{concave term}} \prec 0, \\ & Z = (\hat{E}^{-1})^{-1}. \end{aligned}$$

- $\phi_\infty(\hat{E}^{-1}, Z)$ is concave in $(\hat{E}^{-1}, Z)$.
- The equality constraint is split into two inequality constraints:
  1. $(\hat{E}^{-1})^{-1} - Z \preceq 0 \iff \begin{bmatrix} \hat{E}^{-1} & I \\ I & Z \end{bmatrix} \preceq 0$, which is an **LMI**.
  2. $Z - (\hat{E}^{-1})^{-1} \preceq 0$, which is a **difference of two convex mappings**.

Algorithm 2 (A2): $\mathcal{H}_\infty$-based optimization for fixed edge weights $\hat{W}$

Input: E, L, F, H, Π, $\hat{E}_0$, $\hat{W}$, $\hat{\gamma}_0$, ε

Output: $\hat{E}_*$

Initialize: $k \leftarrow 0$, $(\hat{E}_{(k)}^{-1}, Z_{(k)}) \leftarrow (\hat{E}_0^{-1}, \hat{E}_0)$, and $\hat{\gamma}_{(k)} \leftarrow \hat{\gamma}_0$

While: $\hat{\gamma}_{(k)} - \hat{\gamma}_{(k+1)} \geq \varepsilon$

Solve:

$$\begin{aligned} \min_{\hat{\gamma}, X \succ 0, \hat{E}^{-1}, Z} \quad & \hat{\gamma} \\ \text{s.t.} \quad & \psi_\infty(\hat{\gamma}, X) + \tilde{\phi}_\infty(\hat{E}^{-1}, Z) \prec 0, \\ & \begin{bmatrix} \hat{E}^{-1} & I \\ I & Z \end{bmatrix} \preceq 0, \quad Z - \tilde{f}(\hat{E}^{-1}) \preceq 0, \end{aligned}$$

where $f(X) = X^{-1}$.

Obtain: $\hat{E}_*^{-1}$, Z_*, and $\hat{\gamma}_*$

Update: $k \leftarrow k+1$, $(\hat{E}_{(k)}^{-1}, Z_{(k)}) \leftarrow (\hat{E}_*^{-1}, \hat{E}_*)$, and $\hat{\gamma}_{(k)} \leftarrow \hat{\gamma}_*$

end

$\mathcal{H}_2$-based Optimization

- A similar characterization for $\|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_2}$ is used to formulate optimization problems for the independent selection of edge weights and nodal time-scales of $\hat{\boldsymbol{\Sigma}}$.

$\mathcal{H}_2$-based optimization problem for fixed $\hat{W}$ at iteration k :

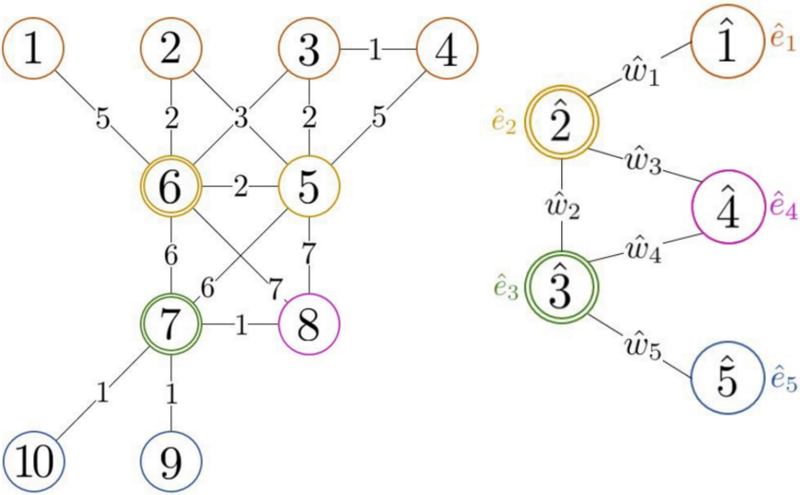
$$\min_{X \succ 0, \hat{E}^{-1}, Z, R} \quad \mathbf{tr}(R)$$

$$\text{s.t.} \quad \psi_2(X) + \tilde{\phi}_2(\hat{E}^{-1}, Z) \prec 0, \;\; Z - \tilde{f}(\hat{E}^{-1}) \preceq 0,$$

$$\begin{bmatrix} X & \hat{\theta}(C + \hat{C})^T \\ * & R \end{bmatrix} \succ 0, \;\; \begin{bmatrix} \hat{E}^{-1} & I \\ * & Z \end{bmatrix} \succeq 0.$$

- All proposed algorithms can be initialized by the solution obtained from applying the Petrov-Galerkin Projection (PGP) paradigm.

Illustrative Example



(a) Original Network (b) Reduced Network

Th following four algorithms are applied:

Algorithm (A1) outputs an $\mathcal{H}_\infty$-suboptimal matrix of weights $\hat{W}_*$ for a given matrix of time-scales $\hat{E}$.

Algorithm (A2) outputs an $\mathcal{H}_\infty$-suboptimal matrix of time-scales $\hat{E}_*$ for a given matrix of weights $\hat{W}$.

Algorithm (A3)* outputs an $\mathcal{H}_2$-suboptimal matrix of weights $\hat{W}_*$ for a given matrix of time-scales $\hat{E}$ **(Cheng et al., 2020)**.

Algorithm (A4) outputs an $\mathcal{H}_2$-suboptimal matrix of time-scales $\hat{E}_*$ for a given matrix of weights $\hat{W}$.

Example Results

Table: Normalized reduction errors using proposed algorithms.

MR Method	$\frac{\|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_\infty}}{\|\boldsymbol{\Sigma}\|_{\mathcal{H}_\infty}}$	MR Method	$\frac{\|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|_{\mathcal{H}_2}}{\|\boldsymbol{\Sigma}\|_{\mathcal{H}_2}}$
PGP	0.146	PGP	0.392
A1	0.076	A3	0.313
A1 $\rightarrow$ A2	0.040	A3 $\rightarrow$ A4	0.049
A2	0.040	A4	0.078
A2 $\rightarrow$ A1	0.039	A4 $\rightarrow$ A3	0.076

Algorithm	Optimized Parameters	
A1	$\hat{e}_* = \begin{bmatrix} 4.0 & 2.0 & 1.0 & 1.0 & 2.0 \end{bmatrix}$	$\hat{w}_* = \begin{bmatrix} 12.5 & 9.5 & 0.8 & 36.4 & 2.0 \end{bmatrix}$
A1 $\rightarrow$ A2	$\hat{e}_* = \begin{bmatrix} 3.7 & 1.5 & 1.6 & 1.5 & 1.9 \end{bmatrix}$	$\hat{w}_* = \begin{bmatrix} 12.5 & 9.5 & 0.8 & 36.4 & 2.0 \end{bmatrix}$

Table: Sample weights and time-scales returned by the proposed algorithms.

Conclusion

- Analyzed the $\mathcal{H}_\infty$ performance for the weighted and time-scaled edge consensus protocol by deriving expressions of and bounds on the $\mathcal{H}_\infty$-norm.
- Derived new insights on minimizing the $\mathcal{H}_\infty$-norm and proposed a versatile optimization setup for the selection of these parameters.
- Proposed $\mathcal{H}_\infty$- and $\mathcal{H}_2$-based model reduction methods for consensus network systems, whereby the parameterized edge weights and nodal time-scales of the reduced consensus network system are tuned via iterative algorithms.
- Improved on existing clustering based approaches by initializing the iterative algorithms using Petrov-Galerkin Projection output.