A Reflexive Tactic for Polynomial Positivity

using Numerical Solvers and Floating-Point Computations

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Joint work with Érik Martin-Dorel

Numerical Optimization

Powerful tool to infer numerical invariants

(x1, x2) $\in \left\{x_1, x_2 \;\middle|\; x_1^2 + x_2^2 \leqslant 1.5^2\right\}$

```
while (1) {
  // Find Inv. p(x1, x2) >= 0
  x1 = x1 * x2;
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}
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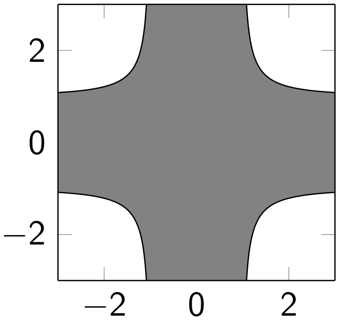

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$$p(x_1, x_2) = 1 + 2.46x_1^2 + 2.46x_2^2 - 5 \times 10^{-7}x_1^4 - 2.46x_1^2x_2^2 - 5 \times 10^{-7}x_2^4$$

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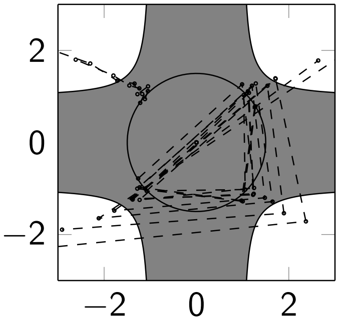

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Can yield incorrect results without warning.

Polynomial Invariants

In a very nice SAS'15 paper, Adjé, Garoche and Magron offer for

$(x1,\ x2) \in [0.9, 1.1] \times [0, 0.2]$

```
while (1) {
  pre_x1 = x1; pre_x2 = x2;
  if (x1^2 + x2^2 <= 1) {
    x1 = pre_x1^2 + pre_x2^3;
    x2 = pre_x1^3 + pre_x2^2;
  } else {
    x1 = 0.5 * pre_x1^3 + 0.4 * pre_x2^2;
    x2 = -0.6 * pre_x1^2 + 0.3 * pre_x2^2;
  }
}
```

the inductive invariant $2.510902467 + 0.0050x_1 + 0.0148x_2 - 3.0998x_1^2 + 0.8037x_2^3 + 3.0297x_1^3 - 2.5924x_2^2 - 1.5266x_1x_2 + 1.9133x_1^2x_2 + 1.8122x_1x_2^2 - 1.6042x_1^4 - 0.0512x_1^3x_2 + 4.4430x_1^2x_2^2 + 1.8926x_1x_2^3 - 0.5464x_2^4 + 0.2084x_1^5 - 0.5866x_1^4x_2 - 2.2410x_1^3x_2^2 - 1.5714x_1^2x_2^3 + 0.0890x_1x_2^4 + 0.9656x_2^5 - 0.0098x_1^6 + 0.0320x_1^5x_2 + 0.0232x_1^4x_2^2 - 0.2660x_1^3x_2^3 - 0.7746x_1^2x_2^4 - 0.9200x_1x_2^5 - 0.6411x_2^6 \geqslant 0.$

Should we trust such results ?

- Some are correct (we'll prove it formally).

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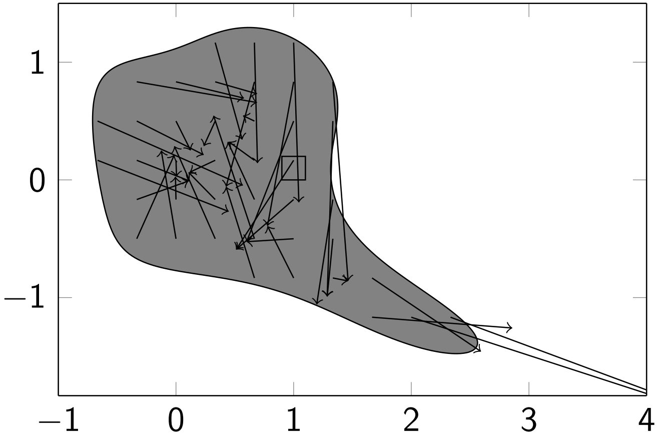
- Some are correct (we'll prove it formally).
- Others aren't (previous degree 6 polynomial)



In Adjé et al. paper

Look for a polynomial p s.t.

$p - \sigma\, q \geqslant 0, \quad \sigma \geqslant 0$	initial condition $(\forall x, q(x) \geqslant 0 \Rightarrow p(x) \geqslant 0)$
$p \circ f - p \geqslant 0$	inductiveness $(\forall x, p(x) \geqslant 0 \Rightarrow p(f(x)) \geqslant 0)$

with $\{x \mid q(x) \geqslant 0\}$ initial set and f loop body.

Then $p \geqslant 0$ is an invariant.

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Need to verify polynomial positivity.

demo.v

Sum of Squares (SOS) Polynomials

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A polynomial p is SOS if there are polynomials $q_1, \ldots, q_m$ s.t.

$$p = \sum_i q_i^2.$$

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- If p SOS then $p \geqslant 0$
- p SOS iff there exist $z := \left[1, x_0, x_1, x_0 x_1, \ldots, x_n^d\right]$ and $Q \succeq 0$ (i.e., for all $x, x^T Q x \geqslant 0$) s.t.

$$p = z^T Q\, z.$$

$\Rightarrow$ SOS can be encoded as semi-definite programming (SDP).

SOS: Example

Example

Is $p(x,y) := 2x^4 + 2x^3y - x^2y^2 + 5y^4$ SOS ?

$$p(x,y) = \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$

that is

$$p(x,y) = q_{11}x^4 + 2q_{13}x^3y + 2q_{23}xy^3 + (2q_{12} + q_{33})x^2y^2 + q_{22}y^4$$

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hence $q_{11} = 2$, $2q_{13} = 2$, $2q_{23} = 0$, $2q_{12} + q_{33} = -1$, $q_{22} = 5$.

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For instance

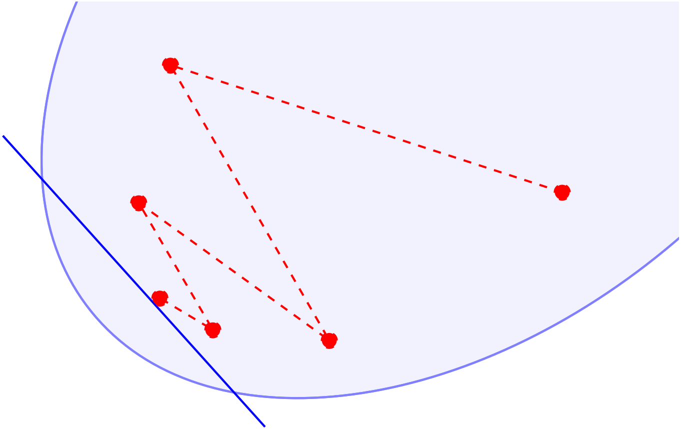
$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L \qquad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

hence $p(x,y) = \frac{1}{2}\left(2x^2 - 3y^2 + xy\right)^2 + \frac{1}{2}\left(y^2 + 3xy\right)^2$.

Inaccuracy in Solving SDPs

SDP solvers only yield approximate solutions due to

- inexact termination



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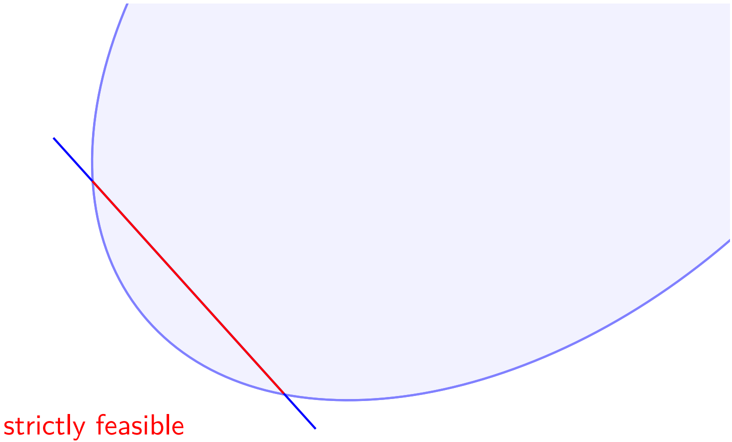

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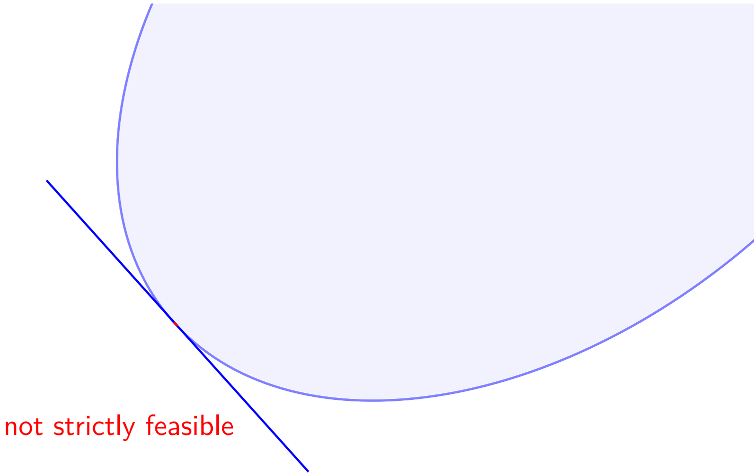

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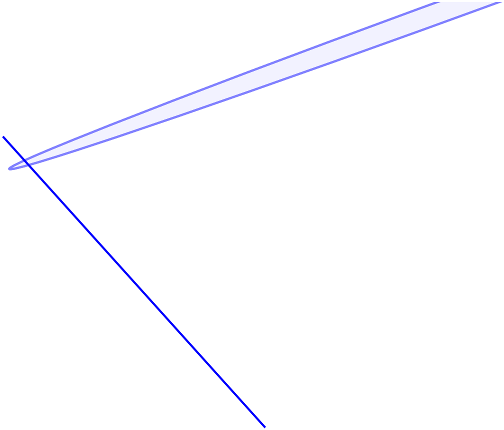

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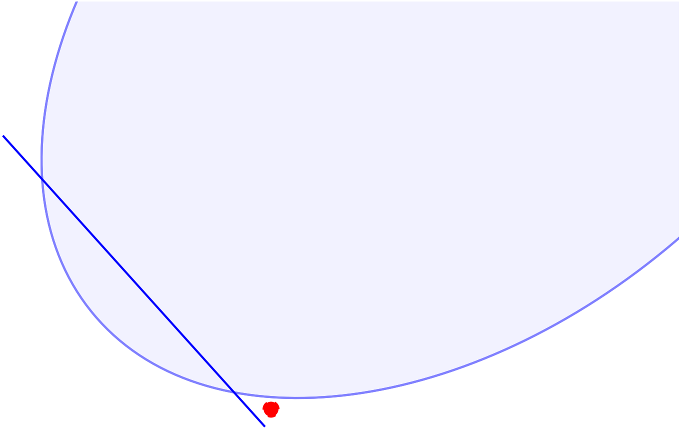

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State of the art [Harrison, Peyrl and Parrilo, Monniaux and Corbineau, Kaltofen et al., Magron et al.]

- round to exact rational solution (heuristic)
- proofs in rational arithmetic (expensive).

SOS: Using approximate SDP solvers

Result Q from SDP solver will only satisfy equality constraints up to some error δ

$$p = z^T Q z + z^T E z, \qquad \forall i\, j, |E_{i,j}| \leqslant \delta.$$

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If $Q + E \succeq 0$ then $p = z^T (Q + E)\, z$ is SOS.

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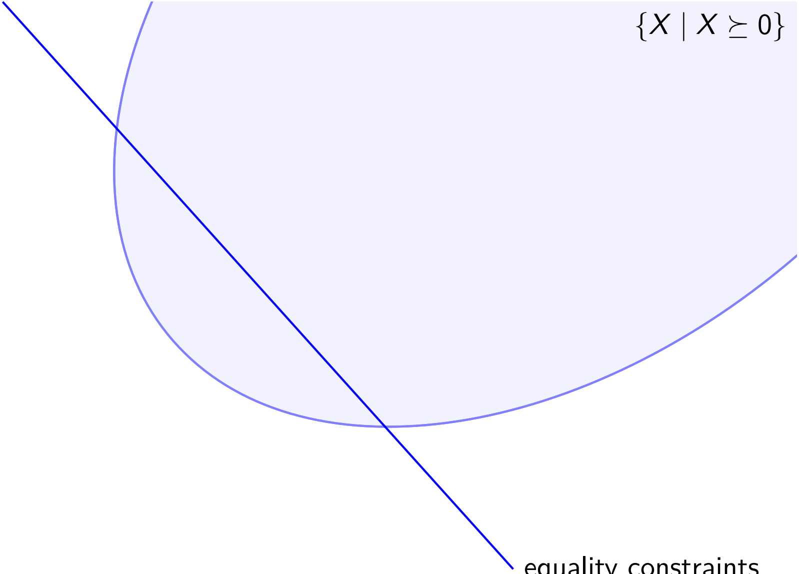
- Hence the validation method: given $p \simeq z^T Q z$
  1. Check that all monomials of p are in $z z^T$.
  2. Bound difference δ between coefficients of p and $z^T Q z$.
  3. If $Q - s \epsilon I \succeq 0$ ($s :=$ size of Q), then p is proved SOS.
- 2 can be done with interval arithmetic and 3 with a Cholesky decomposition ($\Theta(s^3)$ flops).

$\Rightarrow$ Efficient validation method using just floats.

Intuitively



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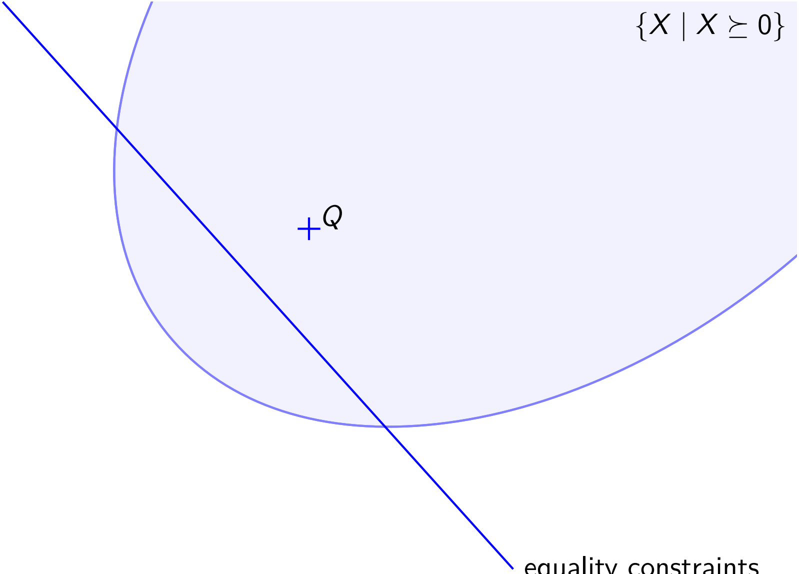

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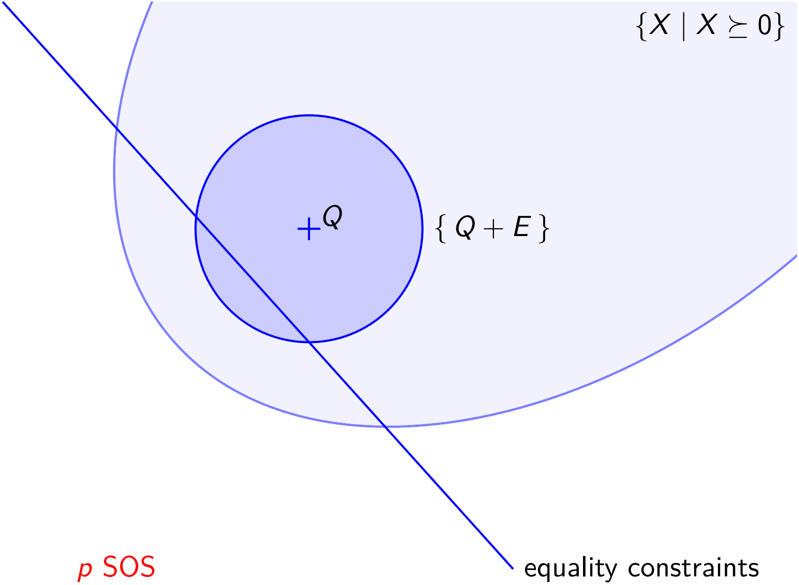

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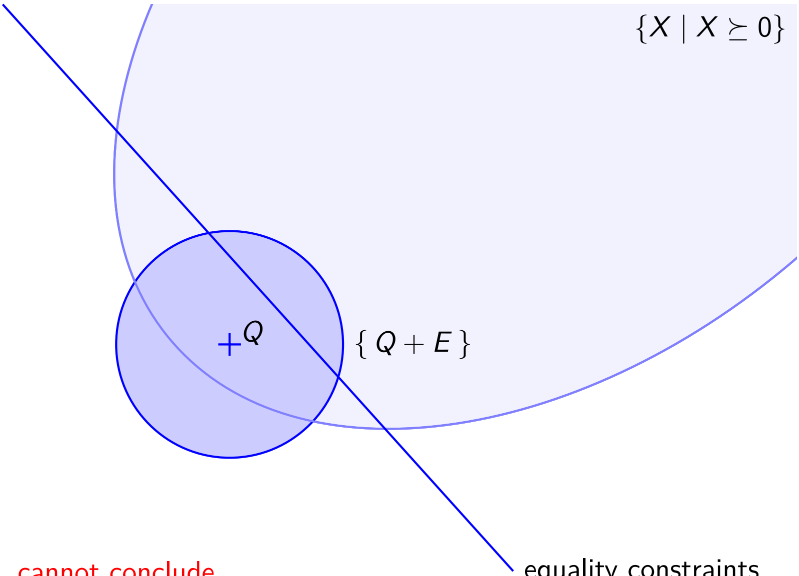

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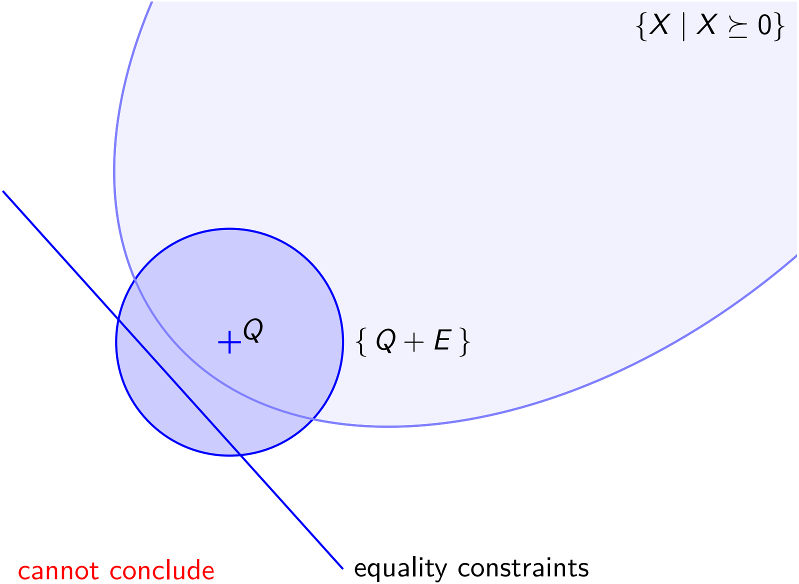

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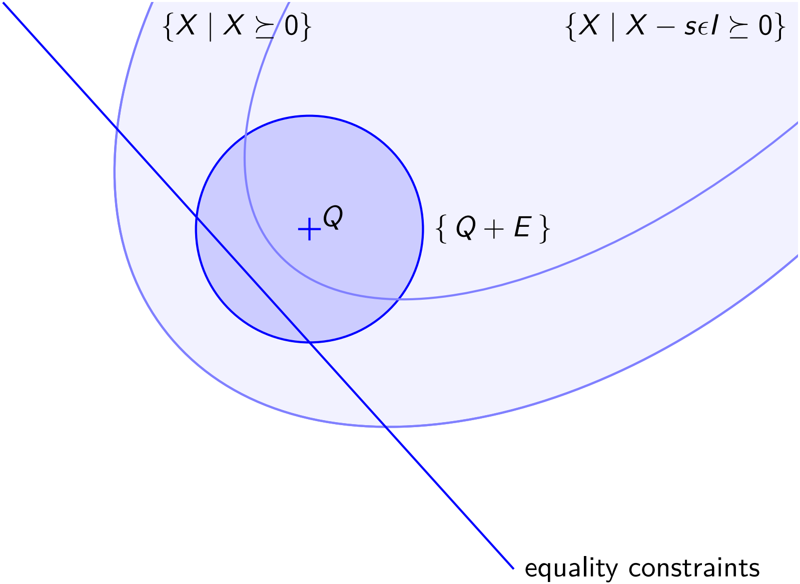
Padding



$\{X \mid X \succeq 0\}$
$\{X \mid X - s\epsilon I \succeq 0\}$
Q
$\{\, Q + E \,\}$
equality constraints

Cholesky Decomposition

- To prove that $a \in \mathbb{R}$ is non negative,
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  (since $x^T \left(R^T R\right) x = (Rx)^T (Rx) = \|Rx\|_2^2 \geqslant 0$).
- The Cholesky decomposition computes such a matrix R:

$$
\begin{array}{l}
R \;\mathbf{:=}\; 0; \\
\textbf{for } j \textbf{ from } 1 \textbf{ to } n \textbf{ do} \\
\quad \textbf{for } i \textbf{ from } 1 \textbf{ to } j-1 \textbf{ do} \\
\qquad R_{i,j} \;\mathbf{:=}\; \left(A_{i,j} - \sum_{k=1}^{i-1} R_{k,i} R_{k,j}\right) / R_{i,i}; \\
\quad \textbf{od} \\
\quad R_{j,j} \;\mathbf{:=}\; \sqrt{M_{j,j} - \sum_{k=1}^{j-1} {R_{k,j}}^2}; \\
\textbf{od}
\end{array}
$$

- If it succeeds (no $\sqrt{}$ of negative or div. by 0) then $A \succeq 0$.

Cholesky Decomposition (end)

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With rounding errors $A \neq R^T R$, Cholesky can succeed while $A \not\succeq 0$.

But error is bounded and for some (tiny) $c \in \mathbb{R}$:
if Cholesky succeeds on A then $A + c\, I \succeq 0$.

Hence:

Theorem

If Cholesky succeeds on $A - c\, I$ then $A \succeq 0$
holds for any $c \geqslant \frac{(s+1)\varepsilon}{1-(2s+2)\varepsilon}\mathrm{tr}(A) + 4(s+1)\left(2(s+2) + \max_i(A_{i,i})\right)\eta$
(ε and η relative and absolute precision of floating-point format).

Proved in Coq (paper proof: 6 pages, Coq: 5.1 kloc)

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   - ⇝ uses SSReflect and MathComp [Gonthier et al.]
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3. Reflexive tactic
   - OCaml code as a wrapper for SDP solvers
   - Some Ltac2 code

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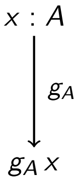

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$$\begin{array}{ccc} x & = & x : A \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g_A} \\ f\,x & = & g_A\,x \end{array}$$

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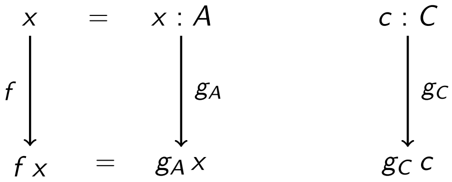

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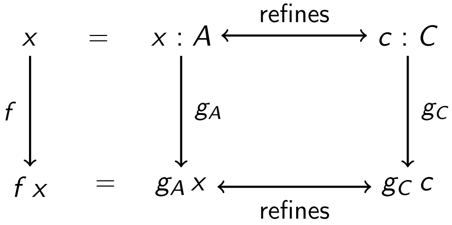

Effective multivariate polynomials

- Implemented in a modular way:

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Definition seqmultinom := list N.
Module MultinomOrd <: OrderedType.
  Definition t := seqmultinom. (*...*) End MultinomOrd.
Module FMapMultipoly (M : Sfun MultinomOrd).
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- Main refinement predicates:

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Rseqmultinom : ∀(n : nat), multinom n → seqmultinom → Type
Reffmpoly : ∀(T : ringType) (n : nat), mpoly n T → effmpoly T
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ReffmpolyC : ∀(A : ringType) (C : Type), (A → C → Type) →
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- Proof-oriented type for coefficients: needs a `ringType` structure; instantiated with MathComp's `rat`.
  Effective counterpart: `bigQ`.

Positive definiteness check for floating-point matrices

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Definition posdef (n : nat) (A : 'M[R]_n) :=
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posdef_check : ∀ (mx : Type → nat → nat → Type)
               (T : Type) (n : nat),
               (*...type classes...→ *)
                mx T n n → bool
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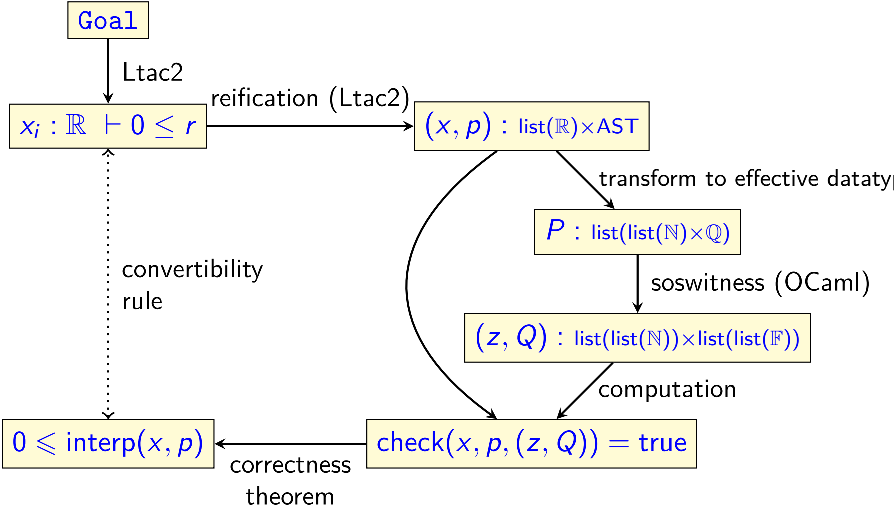
  → or alternatively Coq primitive floats

The validsdp tactic (1/3) – the big picture



The validsdp tactic (2/3) – OCaml code

- Rely on the OSDP lib. (OCaml interface for off-the-shelf SDP solvers)
- Implement a Coq plugin (the `ValidSDP.soswitness` OCaml module provides a `soswitness` tactic that consists of a wrapper for OSDP)

OSDP library: 6.2 kloc of OCaml code + 1.2 kloc of C code.
`ValidSDP.soswitness` plugin: 0.3 kloc of OCaml code.

The validsdp tactic (3/3) – correctness theorem

```
Theorem soscheck_eff_wrapup_correct :
  ∀ (x : list R) (p : p_abstr_poly)
    (zQ : list (list N) * list (list (s_float bigZ bigZ))),
  soscheck_eff_wrapup x p zQ = true →
  (0 ⩽ interp_p_abstr_poly x p)%R.
```

Coq: 2.0 kloc for the main tactic and proofs + 6.5 kloc of refinement proofs (Cholesky: 3.0 kloc; FP arith: 1.3 kloc; multipoly: 2.2 kloc)

OCAML Implementation

- OPAM package: `opam install osdp`
- Available at `https://github.com/Embedded-SW-VnV/osdp`
- LGPL license
- Interface to SDP solvers CSDP, Mosek and SDPA
- 6.2 kloc of OCaml code + 1.2 kloc of C code

Coq Implementation

- OPAM package: `opam install coq-validsdp`
- Available at `https://github.com/validsdp/validsdp`
- LGPL license
- uses libraries
  - CoqEAL [Cano, Cohen, Dénès, Mörtberg, Rouhling, Siles] for refinement proofs (based on SSReflect and MathComp [Gonthier et al.])
  - SSrMultinomials [Strub] for multivariate polynomials
  - CoqInterval [Melquiond] and Flocq [Boldo, Melquiond] for floating-point numbers
- 15 kloc of Coq + 0.3 kloc of OCaml code

Benchmarks (1/3)

Setup:

- A desktop PC under Debian GNU/Linux Jessie
- Core i5-4460S CPU clocked at 2.9 GHz
- All timings are total elapsed time (in seconds)
- Timeout of 900s
- ValidSDP version: d60c663
- library versions: Coq 8.5.2, MathComp 1.6, Flocq 2.5.1, Coquelicot 2.1.1, CoqInterval 3.1.0, OSDP 0.5.2 and dev. version of other libs

Benchmarks (2/3)

Problem	n	d	OSDP (not verified)	MonniauxC11 (not verified)	NLCertify (not verified)	QEPCAD (not verified)	ValidSDP	PVS/Bernstein	NLCertify	HOL Light/Taylor
adaptativeLV	4	4	**0.75**	2.67	1.12	3.29	5.16	14.93	**2.61**	12.31
butcher	6	4	1.58	—	**1.05**	—	9.40	48.44	**8.36**	15.62
caprasse	4	4	**0.41**	1.82	0.88	4.33	5.19	25.89	**2.63**	17.68
heart	8	4	**3.18**	268.75	—	—	**16.67**	131.13	—	26.15
magnetism	7	2	**1.11**	2.04	1.64	4.02	**5.18**	245.52	14.50	16.07
reaction	3	2	0.81	1.56	**0.24**	3.00	4.33	11.48	**1.96**	12.41
schwefel	3	4	**0.95**	2.45	2.76	3.26	**3.70**	14.72	56.13	17.46
fs260	6	4	**1.25**	—	—	—	**5.99**	—	—	—
fs461	6	4	**0.70**	11.18	0.87	—	**5.18**	621.06	7.46	22.70
fs491	6	4	**0.54**	21.81	—	—	**5.38**	—	—	—
fs745	6	4	0.98	11.74	**0.94**	—	**5.55**	623.17	6.90	22.48
fs752	6	2	**0.35**	1.81	0.90	—	**3.80**	54.52	7.88	13.34
fs8	6	2	**0.43**	1.53	1.48	—	**3.93**	52.63	6.62	13.40
fs859	6	8	—	—	—	—	—	—	—	—
fs860	6	4	1.21	10.53	**1.11**	—	**6.08**	73.65	7.34	14.28
fs861	6	4	**1.09**	10.48	1.20	—	**5.15**	69.74	7.87	14.28
fs862	6	4	1.27	79.25	**1.25**	—	**5.37**	73.54	7.58	14.14
fs863	6	2	**0.94**	1.50	—	—	**3.85**	—	—	13.85
fs864	6	2	**0.56**	2.05	—	—	**4.05**	—	—	13.28
fs865	6	2	**0.76**	2.11	—	—	**3.68**	—	—	13.76
fs867	6	2	**0.21**	2.09	1.74	—	**4.22**	—	8.04	—

Benchmarks (3/3)

Problem	n	d	OSDP (not verified)	MonniauxC11 (not verified)	NLCertify (not verified)	QEPCAD (not verified)	ValidSDP	PVS/Bernstein	NLCertify	HOL Light/Taylor
fs868	6	4	**0.94**	—	—	—	**6.05**	—	—	—
fs884	6	4	—	—	—	—	—	—	—	—
fs890	6	4	—	**7.78**	—	—	—	—	—	—
ex4_d4	2	12	—	—	—	—	—	—	—	—
ex4_d6	2	18	—	—	—	—	—	—	—	—
ex4_d8	2	24	**16.99**	—	—	—	**82.89**	—	—	—
ex4_d10	2	30	—	—	—	—	—	—	—	—
ex5_d4	3	8	**1.67**	—	—	—	**13.63**	—	—	—
ex5_d6	3	12	**16.10**	—	—	—	**66.82**	—	—	—
ex5_d8	3	16	**203.06**	—	—	—	**353.70**	—	—	—
ex5_d10	3	20	—	—	—	—	—	—	—	—
ex6_d4	4	8	**16.82**	—	—	—	**44.99**	—	—	—
ex6_d6	4	12	—	—	—	—	—	—	—	—
ex7_d4	2	12	—	—	—	—	—	—	—	—
ex7_d6	2	18	**1.50**	—	—	—	**26.78**	—	—	—
ex7_d8	2	24	**15.38**	—	—	—	**83.47**	—	—	—
ex7_d10	2	30	—	—	—	—	—	—	—	—
ex8_d4	2	8	**0.87**	15.72	—	61.94	**7.52**	—	—	—
ex8_d6	2	12	—	—	—	—	—	—	—	—
ex8_d8	2	16	—	—	—	—	—	—	—	—
ex8_d10	2	20	—	—	—	—	—	—	—	—

ValidSDP (Coq)

invariant.v

Questions

Thanks for your attention!

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Positivstellensatz

We want to prove that

$$p_1(x_1, \ldots, x_n) \geqslant 0 \wedge \ldots \wedge p_m(x_1, \ldots, x_n) \geqslant 0$$

is not satisfiable.

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- equivalence under hypotheses (Putinar's Positivstellensatz)
- no practical bound on degrees of $r_i \Rightarrow$ will be arbitrarily fixed